

# Quiz week 4

1.  $\frac{d}{dx} \left( (1 - 0.8 \sin^2 x) \frac{dy}{dx} \right) - \lambda y = 0 \quad y(0) = y(\pi) = 0$

$$y_0 = y_{N+1} = 0$$

Check if  $\frac{1}{\Delta x^2} \left( p(x_n - \Delta x) y(x_n - \Delta x) - 2p(x_n) y(x_n) + p(x_n + \Delta x) y(x_n + \Delta x) \right)$

$$\approx \frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) \Big|_{x=x_n}$$

$$p(x) = 1 - 0.8 \sin^2 x \quad x = \frac{n\pi}{N+1}$$

Taylor expand.

$$\frac{1}{\Delta x^2} \left( (p(x_n) - \Delta x p'(x_n) + \frac{\Delta x^2}{2} p''(x_n) - \frac{\Delta x^3}{6} p'''(x_n) + \frac{\Delta x^4}{24} p''''(x_n)) \cdot (-\text{II} - \text{for } y) \right)$$

Multiply parts.

$$\begin{aligned} & \frac{1}{\Delta x^2} \left( p(x_n) y(x_n) - \Delta x (p(x_n) y'(x_n) + p'(x_n) y(x_n)) + \frac{\Delta x^2}{2} (p(x_n) y''(x_n) + 2p'(x_n) y'(x_n) + p''(x_n) y(x_n)) \right. \\ & \left. - \frac{\Delta x^3}{6} (p'y^{(3)} + 3p'y'' + 3p''y' + p^{(3)}y) + \frac{\Delta x^4}{24} \underbrace{(py^{(4)} + 4p'y^{(3)} + 6p''y'' + 4p^{(3)}y' + p^{(4)}y)}_{B_n} \right. \\ & \left. + O(\Delta x^5) \right) \end{aligned}$$

Alla negativa termer tas ut av positiva i utvecklingen av  $p(x_n + \Delta x) y(x_n + \Delta x)$

$$\begin{aligned} \text{totalt} \Rightarrow & \frac{1}{\Delta x^2} \left( \frac{\Delta x^2}{2} \cdot 2 (py'' + 2p'y' + p''y) + \frac{\Delta x^4}{24} \cdot 2 \cdot B_n \right) \\ = & py'' + 2p'y' + p''y + \frac{\Delta x^2}{12} B_n \end{aligned}$$

$$\therefore (py')' = p'y' + py'' \neq py'' + 2p'y' + p''y$$

$\Rightarrow$  False

Detta är en diskretisering av

$$\frac{d^2}{dx^2} (py) = \frac{d}{dx} (p'y + py')$$

2. Tridiagonal symmetric toeplitz:

$$T_{\Delta x} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

Skewsymmetric toeplitz:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

lower triangular toeplitz:

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$3. \quad 6u_{j+2} - 5u_{j+1} + u_j = 0 \quad (j=0 : N-1)$$

$$u_0 = 1, \quad u_{N+1} = 0$$

$$6t^2 - 5t + 1 = 0 \iff t^2 - \frac{5}{6}t + \frac{1}{6} = 0$$

$$t = \frac{5}{12} \pm \sqrt{\frac{25}{144} - \frac{24}{144}} = \frac{5}{12} \pm \frac{1}{12} \quad t_1 = \frac{1}{2} \quad t_2 = \frac{1}{3}$$

$$u_j = A \cdot \left(\frac{1}{2}\right)^j + B \left(\frac{1}{3}\right)^j$$

$$u_0 = 1 \Rightarrow 1 = A + B \quad A = 1 - B$$

$$u_{N+1} = 0 \Rightarrow 0 = A \left(\frac{1}{2}\right)^{N+1} + B \left(\frac{1}{3}\right)^{N+1} \iff \left(\frac{1}{2}\right)^{N+1} - B \left(\frac{1}{2}\right)^{N+1} + B \left(\frac{1}{3}\right)^{N+1} = 0$$

$$B = \frac{-\left(\frac{1}{2}\right)^{N+1}}{\left(\frac{1}{3}\right)^{N+1} - \left(\frac{1}{2}\right)^{N+1}} \quad A = 1 - B$$

4.  $\lambda[T]$  egenvärde av  $T$

$\lambda[S]$  egenvärde av  $S$

Gäller för kommuterande matriser, men i allmänhet inte falskt.

5.  $Au = \lambda u$

$$A^{-1}Au = A^{-1}\lambda u$$

$$u = A^{-1}\lambda u$$

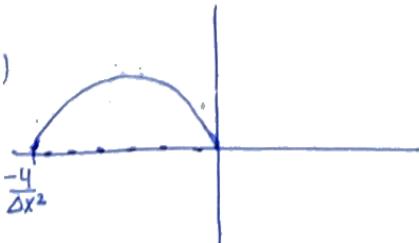
$$A\bar{u} = \frac{1}{\lambda} u$$

6.  $Au = \lambda u \Rightarrow A^2u = A(Au) = \lambda^2 u$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$
$$e^{tA}u = \sum_{k=0}^{\infty} \frac{1}{k!} A^k u = \left( \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k \right) u = e^{\lambda t} u$$

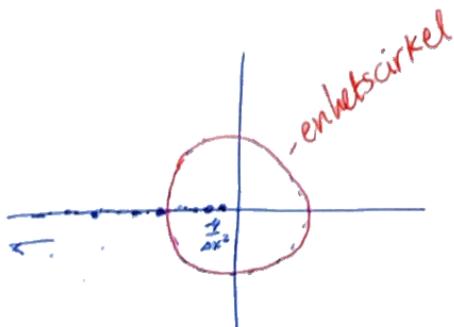
Sant.

7. a.  $\lambda[T_{\Delta x}] = -\frac{4 \sin^2 k\pi}{\Delta x^2} \frac{1}{2(N+1)}$



b.  $T_{\Delta x}^{-1}u = \frac{1}{\lambda_k} u$

$$\lambda[T_{\Delta x}^{-1}] = -\frac{1}{\frac{4 \sin^2(k\pi)}{\Delta x^2} \frac{1}{2(N+1)}} \quad k \neq 0$$



c.  $\Delta x \rightarrow 0 \rightarrow -\frac{1}{\frac{4 \sin^2(k\pi)}{\Delta x^2} \frac{1}{2(N+1)}} \rightarrow -\frac{4 \Delta x^4}{4 k^2 \pi^2} \quad \Delta x \rightarrow 0 \Rightarrow \lambda[T_{\Delta x}^{-1}] \rightarrow 0^-$

$\sin x \approx x$  för små  $x$ .

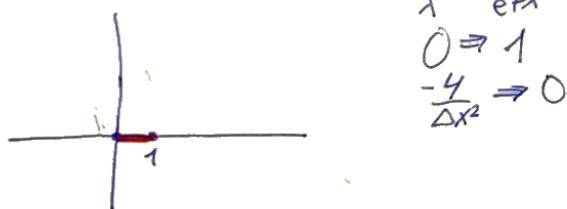
$$8. \dot{u} = T_{\Delta x} u \quad u(0) = v$$

$$a. \|e^{tT_{\Delta x}}\|_2 \quad t > 0$$

$$\|e^{tT_{\Delta x}}\|_2 \leq e^{tM_2[T_{\Delta x}]}$$

b. egenvärde

$$e^{t\lambda}. \quad \lambda = -\frac{4}{\Delta x^2} \sin^2\left(\frac{k\pi}{2(N+1)}\right) \quad \text{Går från } 0 \text{ till } -\frac{4}{\Delta x^2}$$



$$c. \Delta x \rightarrow 0 \Rightarrow$$

$$\text{litet } i\lambda \rightarrow \sin x \rightarrow x$$

$$e^{t\lambda} \rightarrow e^{-t\frac{4}{\Delta x^2} \frac{k^2\pi^2}{4\Delta x^2}} = e^{-k^2\pi^2 t}$$

$$d. t \rightarrow \infty \Rightarrow e^{t\lambda} \rightarrow 0$$

$$e. \|e^{-tT_{\Delta x}}\|_2 \leq -\frac{1}{M_2[T_{\Delta x}]} = -\frac{1}{e^{t\lambda}} \quad \Delta x \rightarrow 0 \rightarrow -\frac{1}{e^{-k^2\pi^2 t}}$$

$$f. \dot{u} = T_{\Delta x} u$$

$$u_{n+1} = u_n + \Delta t T_{\Delta x} u_n$$

$$\rightarrow |1 + \Delta t T_{\Delta x}| \leq 1$$

$$g. u_{n+1} = u_n + \frac{\Delta t}{2} T_{\Delta x} u_{n+1}$$

$$u_{n+1} = \frac{1}{1 - \frac{\Delta t}{2} T_{\Delta x}} u_n \quad \left| \frac{1}{1 - \frac{\Delta t}{2} T_{\Delta x}} \right| \leq 1$$

$$h. \Delta x \rightarrow 0 \quad T_{\Delta x} \rightarrow \infty \quad |1 + \Delta t \infty| \text{ större än } 1$$

$$\left| \frac{1}{1 - \frac{\Delta t}{2} \infty} \right| \rightarrow 0 \text{ och är bättre}$$