

Quiz week 3

1. $\mu[A] = \sup_{x \neq 0} \frac{\operatorname{Re} \langle x, Ax \rangle}{\|x\|^2} = \lim_{h \rightarrow 0^+} \frac{\|I + hA\| - 1}{h}$

2. True

3. $\|A^{-1}\|_\infty \leq \frac{1}{\mu_0[A]}$

if $\mu[A] < 0$ then A is nonsingular and

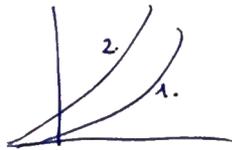
$$\|A^{-1}\|_\infty \leq \frac{1}{\mu_0[A]}$$

4. 1. $\|x(t)\| \leq e^{t\mu[A]} \|x(0)\|$; $t \geq 0$
 2. $\|x(t)\| \leq e^{t\|A\|} \|x(0)\|$; $t \geq 0$

$$\mu[A] \leq \|A\|$$

$$e^{t\mu[A]} \leq e^{t\|A\|}$$

1 is sharper.



5. $A = \lambda$ complex $\|A\| = |\lambda|$

$$\operatorname{Re}(\lambda) = a \quad \operatorname{Im}(\lambda) = b$$

$$\mu[A] = \lim_{h \rightarrow 0^+} \frac{\|I + hA\| - 1}{h} = \lim_{h \rightarrow 0^+} \frac{\|I + h\lambda\| - 1}{h} = \lim_{h \rightarrow 0^+} \frac{\sqrt{(1+h\lambda)^2} - 1}{h}$$

$$\lim_{h \rightarrow 0^+} \frac{(\sqrt{(1+ha)^2 + (hb)^2} - 1) / (\sqrt{(1+ha)^2 + (hb)^2} + 1)}{h} = \lim_{h \rightarrow 0^+} \frac{(1+ha)^2 + (hb)^2 - 1}{h(\sqrt{(1+ha)^2 + (hb)^2} + 1)}$$

$$\lim_{h \rightarrow 0^+} \frac{1 + 2ha + h^2a^2 + h^2b^2 - 1}{h(\sqrt{(1+ha)^2 + h^2b^2} + 1)} = \lim_{h \rightarrow 0^+} \frac{2a + ha^2 + hb^2}{\sqrt{(1+ha)^2 + h^2b^2} + 1} = \frac{2a}{2} = a = \operatorname{Re}(\lambda)$$

$$\mu[A] = \operatorname{Re}(\lambda)$$

$$6. \quad \lim_{h \rightarrow 0^+} \frac{|1+hz|-1}{h}$$

Ersätt A med z i uppgift 5, där z är komplex tal med $\operatorname{Re}(z) = a$ och $\operatorname{Im}(z) = b$.

$$\rightarrow \lim_{h \rightarrow 0^+} \frac{|1+hz|-1}{h} = \operatorname{Re}(z)$$

$$7. \quad \dot{y} = \lambda y, \quad y(0) = 1$$

$$|y(t)| \leq e^{t|\lambda|} \quad \text{Sharp bound?}$$

Krav: y är övergräns om $y \geq x$ för alla x och om det existerar $x_0 \in X$ så att $y = x_0$.

$$|y(t)| \leq e^{t|\lambda|} \Rightarrow \ln|y(t)| \leq t|\lambda|$$

$$|\lambda| \geq \frac{\ln|y(t)|}{t}$$

$$\mu[A] = \operatorname{Re}|\lambda|$$

$$|y(t)| \leq e^{t \operatorname{Re}(\lambda)} \Rightarrow \operatorname{Re}(\lambda) \geq \frac{\ln|y(t)|}{t}$$

8.

	E	Δ	∇	M	W	hD
E	X	$\Delta+1$	$(1-\nabla)^{-1}$	$2M-1$	$(2W-1)^{-1}$	e^{hD}
Δ	$E-1$	X	$(1-\nabla)^{-1}-1$	$2M-2$	$(2W-1)^{-1}-1$	$e^{hD}-1$
∇	$1-E^{-1}$	$1-(\Delta+1)^{-1}$	X	$1-(2M-1)^{-1}$	$2W$	$1-e^{-hD}$
M	$\frac{E+1}{2}$	$\frac{\Delta+2}{2}$	$\frac{(1-\nabla)^{-1}+1}{2}$	X	$\frac{(2W-1)^{-1}+1}{2}$	$\frac{e^{hD}+1}{2}$
W	$\frac{1+E^{-1}}{2}$	$\frac{1+(\Delta+1)^{-1}}{2}$	$\frac{1+(1-\nabla)^{-1}}{2}$	$\frac{1+(2M-1)^{-1}}{2}$	X	$\frac{1+e^{-hD}}{2}$
hD	$\ln(E)$	$\ln(\Delta+1)$	$-\ln(1-\nabla)$	$\ln(2M-1)$	$-\ln(2W-1)$	X

$$E = e^{hD} = y(t) \mapsto y(t+h)$$

$$M = (E+1)/2$$

$$E^{-1} = e^{-hD}$$

$$\Delta = E-1$$

$$\nabla = 1-E^{-1}$$

$$W = \frac{1+E^{-1}}{2}$$

$$D = y \mapsto \dot{y} = \frac{d}{dt}$$

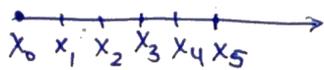
$$9. \quad y'' = x^2 + y^2 \quad y(0) = y(1) = 0$$

$$y'' = \frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2}$$

$$\frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} = y_n^2 + x_n^2$$

$$F(y) = 0 \Rightarrow \frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} - x_n^2 - y_n^2 = 0$$

$$N = 4$$



$$\Delta x = \frac{1}{5} = \frac{1}{N+1}$$

$$y_0 = 0, y_5 = 0$$

$$F(y_1) = 0 \Rightarrow \frac{y_2 - 2y_1}{(0.2)^2} - 0.2^2 - y_1^2 = 0$$

$$F(y_2) = 0 \Rightarrow \frac{y_1 - 2y_2 + y_3}{0.2^2} - 0.4^2 - y_2^2 = 0$$

$$F(y_3) = 0 \Rightarrow \frac{y_2 - 2y_3 + y_4}{0.2^2} - 0.6^2 - y_3^2 = 0$$

$$F(y_4) = 0 \Rightarrow \frac{y_3 - 2y_4}{0.2^2} - 0.8^2 - y_4^2 = 0$$

$$10. \quad f'(x) = \left\{ \frac{\partial f_i}{\partial y_i} \right\} = \begin{pmatrix} -\frac{2}{\Delta x^2} - 2y_1 & \frac{1}{\Delta x^2} & 0 & 0 \\ \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} - 2y_2 & \frac{1}{\Delta x^2} & 0 \\ 0 & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} - 2y_3 & \frac{1}{\Delta x^2} \\ 0 & 0 & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} - 2y_4 \end{pmatrix}$$

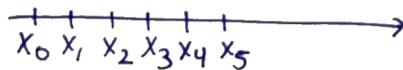
- 11.
1. Beräkna Jacobian. ✓
 2. Faktorisera Jacobian $F'(y^k) \rightarrow LU$
 3. Lös linjärt system $LU \delta y^k = -F(y^k)$
 4. Uppdatera $y^{k+1} = y^k + \delta y^k$

12. $y'' = x^2 + y^2 \quad y(0) = y'(1) = 0$

$$y'' = \frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} \quad N=4$$

$$\Delta x = \frac{1}{N} = \frac{1}{4}$$

$$x_0 = 0 \quad x_4 = 1 \quad x_5 = 1 + 0,25 = 1,25$$



$$y_0 = 0$$

$$F(y_1) = 0 \Rightarrow \frac{-2y_1 + y_2}{0,25^2} - 0,25^2 - y_1^2 = 0$$

$$F(y_2) = 0 \Rightarrow \frac{y_1 - 2y_2 + y_3}{0,25^2} - 0,5^2 - y_2^2 = 0$$

$$F(y_3) = 0 \Rightarrow \frac{y_2 - 2y_3 + y_4}{0,25^2} - 0,75^2 - y_3^2 = 0$$

$$y' = \frac{y_{n+1} - y_{n-1}}{2\Delta x} \quad y' = \beta \Rightarrow \frac{y_{n+1} - y_{n-1}}{2\Delta x} = \beta$$

$$y_{n+1} = 2\beta\Delta x + y_{n-1}$$

$$F(y_4) = 0 \Rightarrow \frac{2\beta \cdot 0,25 + 2y_3 - 2y_4}{0,25^2} - 1 - y_4^2 = 0$$

$$F(y_5) = 0 \Rightarrow 2\beta \cdot 0,25 + y_4$$