Runge-Kutta methods

$$\begin{cases} Y_i = y_n + \Delta t \sum\limits_{j=1}^{S} a_{ij} Y_j' \\ Y_i' = f(t_n + c_i \Delta t, Y_i) \\ y_{n+1} = y_n + \Delta t \sum\limits_{j=1}^{S} b_j Y_j' \end{cases}$$

Butcher Tableaus

Implicit: $\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$ Explicit: $\begin{array}{c|ccc} & 0 & \cdots & 0 \\ c & & \ddots & \vdots \\ & \tilde{A} & & 0 \\ \hline & & b^T & \end{array}$ SDIRK: $\begin{array}{c|ccc} & \gamma & \cdots & 0 \\ c & & \ddots & \vdots \\ & \tilde{A} & & \gamma \\ \hline & & b^T & \end{array}$

- Order: Taylor expand
- Stability:
  $$R(z) = \frac{det(I - zA + zeb^T)}{det(I - zA)}$$
- ERK cannot be A-stable. IRK A-stable iff $R(z)$ has all poles in rhs plane and $|R(z)| \le 1$ on the imaginary axis

Adams methods

Adams-Bashforth (ABk)

$$\begin{cases} y_{n+1} = y_n + \Delta t \sum\limits_{i=1}^{k} \beta_{k+1-i} f_{n+1-i} \\ \beta_{k+1-i} = \int\limits_0^1 l_{k+1-i}(t_{n+1-i} + s\Delta t) ds \\ l_{k+1-i}(t) = \prod\limits_{j=1, j\neq i}^{k} \dfrac{t - t_{n+1-j}}{t_{n+1-i} - t_{n+1-j}} \end{cases}$$

- Order: k
- Stability: not A-stable for k > 2
- Explicit
- AB1 = Explicit Euler

Adams-Moulton (AMk)

$$\begin{cases} y_{n+1} = y_n + \Delta t \sum\limits_{i=0}^{k} \beta_{k-i} f_{n+1-i} \\ \beta_{k-i} = \int\limits_0^1 l_{k+1-i}(t_{n+1-i} + s\Delta t) ds \\ l_{k+1-i}(t) = \prod\limits_{j=0, j\neq i}^{k} \dfrac{t - t_{n+1-j}}{t_{n+1-i} - t_{n+1-j}} \end{cases}$$

- Order: k + 1
- Stability: not A-stable for k > 1
- Implicit
- AM0 = Implicit Euler

BDF methods

$$\begin{cases} p_{k+1}(t) = \sum\limits_{i=0}^{k} y_{n+1-i} l_{k+1-i}(t) \\ l_{k+1-i}(t) = \prod\limits_{j=0, j\neq i}^{k} \dfrac{t - t_{n+1-j}}{t_{n+1-i} - t_{n+1-j}} \\ p_{k+1}'(t_{n+1}) = f_{n+1} \end{cases}$$

Idea: build $p(t)$, differentiate, insert $t = t_{n+1}$, rearrange to isolate y_{n+1}.

- Order: k
- Stability: not A-stable for k > 2
- Implicit
- BDF1 = Implicit Euler

Definitions

- Local error: $l_{n+1} = y(t_{n+1}) - \hat{y}_{n+1}$; $\hat{y}_{n+1}$ is the numerical solution after one step if y_n is exact.
- Global error: $e_{n+1} = y(t_{n+1}) - y_{n+1}$
- Convergence: error $\to 0$ as $\Delta t \to 0$
- Order of Consistency: p if $||l_{n+1}|| = O(\Delta t^{p+1})$
- Order of Convergence: p if $||e_{n+1}|| = O(\Delta t^p)$
- Stability region: The set $D \subset \mathbb{C}$ s.t. the method gives bounded solutions to $y' = \lambda y$, $\text{Re}(\lambda) \le 0$.
- A-stability: If $\mathbb{C}^- \subset D$.
- Lipschitz continuous: if L exists s.t. $||f(u) - f(v)|| \le L||u - v||$
- Finite escape time: solution blows up (not Lipschitz continuous)

Solving Implicitness

Predictor-Corrector

Use an explicit method to predict y_{n+1}, use FPI or Newton to iteratively correct. PE(CE)$^\infty$ $\to$ PE(CE)$^\infty$.

Fixed Point Iteration

$$\begin{cases} y = G(y) \\ y^{(i+1)} = G(y^{(i)}) \end{cases}$$

- Idea: G is the "rhs" of your method. Keep inserting until convergence.
- Pro: Computationally inexpensive.
- Con: Requires a certain time step.

Ex: Implicit Euler

$G(y) = y_n + \Delta t f(t_{n+1}, y)$

Newton's Method

$$\begin{cases} F(y) = 0 \\ F'(y)\Delta y = -F(y) \\ y^{(i+1)} = y^{(i)} + \Delta y \end{cases}$$

- Idea: F is y minus the "rhs" of your method.
- Pro: Guarantees convergence, no restriction on time step.
- Con: Computationally expensive.
- Simplified Newton: Only compute the Jacobian at the start of each step.

Ex: Implicit Euler

$$\begin{cases} F(y) = y - y_n - \Delta t f(t_{n+1}, y) \\ F'(y) = I - \Delta t f_y'(t_{n+1}, y) \end{cases}$$

Finite Differences

$$\begin{cases} -u''(x) + u(x) = 0 \quad 0 \le x \le 1 \\ u(0) = \alpha, u(1) = \beta \end{cases}$$

1. Introduce spatial grid, $\Delta x = \frac{1}{M+1}$
2. Discretize:
$$\begin{cases} u(x_m) \mapsto U_m \\ u''(x_m) \mapsto \dfrac{U_{m-1} - 2U_m + U_{m+1}}{\Delta x^2} \end{cases}$$
3. Write the discretized equation as:
$$\begin{cases} -\dfrac{-2U_1 + U_2}{\Delta x^2} + U_1 = \dfrac{\alpha}{\Delta x^2} \\ -\dfrac{U_{m-1} - 2U_m + U_{m+1}}{\Delta x^2} + U_m = 0 \\ -\dfrac{U_{M-1} - 2U_M}{\Delta x^2} + U_M = \dfrac{\beta}{\Delta x^2} \end{cases}$$
4. Write this in matrix form:
$$\left(I - \frac{1}{\Delta x^2} T_{\Delta x}\right) U = c_{\Delta x}$$

with

$$T_{\Delta x} = \begin{bmatrix} -2 & 1 & 0 \dots & 0 \\ 1 & -2 & 1 \dots & 0 \\ \vdots & \vdots & \vdots \ \ddots & \vdots \\ 0 & 0 & 0 \dots & -2 \end{bmatrix}, c_{\Delta x} = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ \beta \end{bmatrix}$$

$T_{\Delta x}$ and $c_{\Delta x}$ depend on the BVP. For the $T_{\Delta x}$ above, its eigenvalues are $\lambda_m = -4 \sin\left(\dfrac{\pi m}{2(M+1)}\right)$

With Neumann or Robin conditions, you need to discretize the derivative. This can be done in 3 ways:

$$u'(x_m) \mapsto \begin{cases} \dfrac{U_{m+1} - U_m}{\Delta x} & \text{order 1} \\ \dfrac{U_m - U_{m-1}}{\Delta x} & \text{order 1} \\ \dfrac{U_{m+1} - U_{m-1}}{2\Delta x} & \text{order 2} \end{cases}$$

$V = \left[u(t_e, x_1), \dots, u(t_e, x_M)\right]^T \Rightarrow$ $||V - U||_{2,\Delta x} \le \frac{\Delta x^2}{12} \max |\frac{d^4}{dx^4} u(x)|$

Nonlinear BVP

$$\begin{cases} -u''(x) = f(x, u(x)) \quad 0 \le x \le 1 \\ u(0) = \alpha, u(1) = \beta \end{cases}$$

Solve with Newton's method;

$$F(U) = -\frac{1}{\Delta x^2} T_{\Delta x} U - \begin{bmatrix} f(x_1, U_1) \\ \vdots \\ f(x_M, U_M) \end{bmatrix} + c_{\Delta x}$$

$$F'(U) = -\frac{1}{\Delta x^2} T_{\Delta x} - \begin{bmatrix} f_1' & 0 & \cdots & 0 \\ 0 & f_2' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_M' \end{bmatrix}$$

Method of Lines

Idea: Begin with PDE. Discretize in space using Finite Differences $\Rightarrow$ IVP. Use an IVP method (eg. Runge-Kutta) $\Rightarrow$ Recursion over linear systems.

$$\begin{cases} \frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = 0 \\ u(t, 0) = u(t, 1) = 0 \\ u(0, x) = g(x) \end{cases} \mapsto \begin{cases} \frac{d}{dt} U - \frac{1}{\Delta x^2} T_{\Delta x} = 0 \\ U(0) = V_0 \end{cases} \mapsto \begin{cases} U_{n+1} = R_{\Delta x} U_n \\ U_0 = V_0 \end{cases}$$

Linear Spaces

$||x||$ is a norm if

- $||x|| \ge 0$ and $||x|| = 0 \Leftrightarrow x = 0$
- $||\lambda x|| = |\lambda| \cdot ||x||$
- $||x + y|| \le ||x|| + ||y||$

$||x||_{2,c} = \left(c \sum\limits_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} = (c \cdot x^T x)^{\frac{1}{2}}$

$||u||_{L^2} = \left(\int\limits_a^b u(x)^2 dx\right)^{\frac{1}{2}}$

Matrix norm: $||A|| = \sup\limits_{x \neq 0} \dfrac{||Ax||}{||x||}$

$||Ax|| \le ||A|| \cdot ||x||$

- A orthogonal: $||A|| = 1$
- A symmetric: $||A|| = \max\limits_k |\lambda_k[A]|$

Method of Lines

- $R_{\Delta x}^{EE} = \left(I + \frac{\Delta t}{\Delta x^2} T_{\Delta x}\right)$
- $R_{\Delta x}^{IE} = \left(I - \frac{\Delta t}{\Delta x^2} T_{\Delta x}\right)^{-1}$
- $R_{\Delta x}^{TR} = \left(I - \frac{\Delta t}{2\Delta x^2} T_{\Delta x}\right)^{-1} \left(I + \frac{\Delta t}{2\Delta x^2} T_{\Delta x}\right)$

Stability

- $U_{n+1} = R_{\Delta x} U_n$ is stable if $||R_{\Delta x}^n|| < C$, C independent of n and Δx.
- CFL condition: $\frac{\Delta t}{\Delta x^2} \le \frac{1}{2} \Rightarrow$ Explicit Euler is stable.
- IE and TR are stable.

Convergence

$V = \left[u(t_e, x_1), \dots, u(t_e, x_M)\right]^T$.

- $||V_N - U_N^{EE}||_{2,\Delta x} \le C(\Delta t + \Delta x^2)$
- $||V_N - U_N^{IE}||_{2,\Delta x} \le C(\Delta t + \Delta x^2)$
- $||V_N - U_N^{TR}||_{2,\Delta x} \le C(\Delta t^2 + \Delta x^2)$

For EE & IE:
lhs $\le t_e(\frac{\Delta t}{2} \max |\frac{\partial^2}{\partial t^2} u| + \frac{\Delta x^2}{12} \max |\frac{\partial^4}{\partial x^4} u|)$

PDEs

Diffusion	$\frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = f$
Wave	$\frac{\partial^2}{\partial t^2} u - \frac{\partial^2}{\partial x^2} u = f$
Poisson	$-\frac{\partial^2}{\partial x^2} u - \frac{\partial^2}{\partial y^2} u = f$
Schrödinger	$i \frac{\partial}{\partial t} u + \frac{\partial^2}{\partial x^2} u = Vu$

Time Step Adaptivity

Goal: $||l_{n+1}|| = TOL$

$\Delta t_{n+1} = \Delta t_n \left(\dfrac{TOL}{||l_{n+1}||}\right)^{\frac{1}{p}}$

Two ways to estimate l_{n+1}:

- Use time steps of different sizes
- Use different order methods. For RK methods, use embedded RK

Embedded RK: $\begin{array}{c|c} c & A \\ \hline & b \\ & \hat{b} \end{array}$

$\hat{b}$ gives an RK of lower order.
$||\hat{l}_{n+1}|| = ||\hat{y}_{n+1} - y_{n+1}||$ can be used for error estimation.

- RK stability: $R(z) = 1 + \mathbf{b}^T(I - zA)^{-1}\mathbf{1}$
- tänk att det står i i nedersta raden på RK
- Max order of consistency for RK is s, at least for $s \leq 4$
- Stability for EE: Apply to linear test function and arrive at $y_{n+1} = (1 + h\lambda)y_n$. Put $z = \lambda h$ and check where $|\frac{y_{n+1}}{y_n}| = R(z) = |1 + z| < 1$. This becomes a circle with radius 1 centered in -1, which puts a condition on z and therfore on h
- Root condition: We say that a linear k-step method satisfies the root condition if the roots of the characteristic polynomial all lie within or on the unit circle, those on the unit circle being simple.
- Zero stability: A a linear k-step method is zero stable if and only if it satisfies the root condition.
- No method that is not zero-stable can be A-stable.
- a method that is explicit can not be A-stable.
- If a method is A-stable, there is no longer any restrictions on the timestep Δt.
- Karaktäristisk polynom för saker: ställ upp y_m-termer i VL och $f(t_m, y_m)$ i HL. Få högsta y-termen fri
- Taylor
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
Ex, flerdimtaylor
$$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \tfrac{1}{2!}(f_{xx}(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(y-b)^2)$$
- Trapezoidal:
$$y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1}))$$
- Vid Neumann $\Delta x = \frac{1}{M+1/2}$, sätt $\beta = (U_{M+1} - U_M)/\Delta x$
- Jacobian:
$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$
- Dahlquists first barrier theorem: The maximal order of a zero-stable k-step method is k if it is a explicit method, $k+1$ if k is odd and the method is implicit and $k+2$ if k is even and the method is implicit.
- Dahlquist second barrier theorem: There are no explicit A-stable and linear multistep methods. The implicit ones have order of convergence at most 2. The trapezoidal rule has the smallest error constant amongst the A-stable linear multistep methods of order 2.
- BDF- Backward Differntial Formula (family of implicit methods), not A-stable for k>2 and not zero-stable for k>6
- Picard-Lindelöf: if $f(t,y)$ is continous on an intervall in t and Lipschitz continous with $L[f]$ independent of t, then there exists a unique solution of the IVP on the interval for every $y(0) = y_0$.
- Lemma Assume $u \in C^4(a,b)$, then:
$$\frac{u(x_{m+1} - 2u(x_m) + u(x_{m-1})}{\Delta x^2} = u''(x_m) + \Delta x \frac{-1}{12} u^{(4)}(\epsilon)$$
where $\epsilon \in [x_{m-1}, x_{m+1}]$. (This expression can be used to approximate the second derivative.)Proof: Taylor expand around $x = x_n$.
- Theorem: Let u be a solution of $u\mathtt{tt}(x) + u(x) = 0, u(0) = \alpha u(1) = \beta$ and U its finite difference approximations given by
$$(I - (1/\Delta x^2)T_{\Delta x})U = c_{\Delta x}$$
then:
$$||u(x_1), ..., u(x_m) - U||_{2,\Delta x} \leq \Delta x^2 \frac{1}{12} max_{x\in[0,1]} |\frac{d^4}{dx^(4)} u(x)|$$
- Matrices
  - A diagonal $\rightarrow ||A|| = max_k |\lambda_k|$ (Use that $||A||_2 = x^T A^T A x = \sum_{k=1}^n \lambda_k^2 x_k^2$)
  - A orthognal $\rightarrow ||A|| = 1$ (Orthogonal=normailized and pairwise orthogonal, $A^T A = I$)
  - A symmetric ($A = A^T$)$\rightarrow ||A|| = max_k |\lambda_k|$ (Use SDS^T, S orthogonal and D diagonal.)
- Stability of MOL: DEF: A full discretisation scheme $U^{n+1} = R_{\Delta x} U^n$ is stable if $||R^n_{\Delta x}|| \leq C$ with C being independent of both n and Δx.Ex with EE:
$$||R^{EE}_{\Delta x}||_2 = ||S(I + \tfrac{\Delta t}{\Delta x^2} D)S^T||_2 \leq ||S||_2 (I + \tfrac{\Delta t}{\Delta x^2} D)||_2 ||S^T||_2 =$$
$$||I + \tfrac{\Delta t}{\Delta x^2} D|| = max_{1<m<M} |1 - 4\tfrac{\Delta t}{\Delta x^2} sin^2(m\pi/(2(M+1)))| \leq |1 - 4\tfrac{\Delta t}{\Delta x^2}|$$
We will have stability with $||R^{EE}_{\Delta x}||_2 < 1$ if $\frac{t}{\Delta x^2} \leq 1/2$
- The condition on Δt and Δx is called the CFL-condition. For different equations the constant $1/2$ may change, but the format stays the same (se ex nedan). The implicit methods have better stability properties, and the implicit Euler / finite dif. scheme is stable without any restrictions on Δt or Δx (at least for the case we have studied, but presumably in general as well).
- For the diffusion eq $u'_t = a u''_{xx}$ with $a = pos.const.$ the CFL condition becomes $a \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$
- The eigenvalues of $T_{\Delta x}$ is $\lambda_{m,T_{\Delta x}} = -4 sin^2(\frac{\pi m}{2(M+1)})$.
Proof: $T_{\Delta x} = -2I + S$, där S har nollor på diagonalen och ettor på subdiagonalerna. Hence, $\lambda_{m,T_{\Delta x}} = -2 + \lambda_m[S]$. Wee need to solve $Sx = \lambda x$, or the equivalent linear differnece eq:
$$x_{m+1} - \lambda_m + x_{m-1} = 0, x_0 = x_{M+1} = 0$$
which gives the characteristic pol. $z^2 - \lambda z + 1 = 0$. Let z_{+-} be the two roots, $z_\pm = \lambda/2 \pm \sqrt{1 - \lambda^2/4}$. Case1: $z_+ = z_-$ ($\lambda = 2, z_\pm = 1$). Then $x_m = Az_+^m + Bmz_-^m = A + Bm$. With the boundary values this gives $x_m = 0$ for every m.
Case 2: $x_m = Az^{m_+ + B_{z}-^m}$. Note that $(z - z_+)(z - z_-) = z^2 + (z_+ + z_-)z + z_+ z_- = z^2 - \lambda z + 1$ which gives us $z_- = 1/z_+$ and $x_m = Az_+^m + Bz_-^{-m}$. $x_0 = A + B \implies x_m = A(z_+^m - z_-^{-m})$. Vidare: $x_{M+1} = 0 \iff z_+^{M+1} = z_+^{-(M+1)}$. Förläng med $z_+^{-(M+1)}$ så blir $z_+^{2(M+1)} = 1 = e^{2\pi i} \implies {}_{z_+ \neq 1} z_+ = e^{\pi i/(M+1)}$. Now, $= z_+ + z_-$ (med litet m med). Skriv om till $2cos(...)$. $\lambda_{m,T_{\Delta x}} = -2 + 2cos(...)$. Utnyttja $cos(2x) = 1 - 2sin^2(x)$.
- The eigenvalues for a non-symmetric tridiagonal $n \times n$ Toeplitz matriz $A = tridiag(b\ a\ c)$ are $\lambda_k = a + 2\sqrt{bc} cos(\frac{k\pi}{N+1}),\ k = 1:N$.
- Does $Au = (I - (1/\Delta x^2)T_{\Delta x})u$ have a unique solution? Yes beacuse $\lambda_A = I - \lambda_{T\Delta x} > 0$.
- After discretising $u'_t = \frac{1}{Pe} u''_{xx} + u'_x$ with $u(t,0) = u(t,1) = 0$ and $u(0,x) = g(x)$ we, with $u_j(t) = u(t, x_j)$, arrive at
$$u'_t = Au = (\frac{1}{Pe} T_{\Delta x} + S_{\Delta x})u,$$
with initial condition $u_j(0) = g(x_j)$. This is an IVP (in fact n IVP:s). Apply method. The apply method step is really a discretisation in t, with $\Delta t = t_{end}/N$. The equation (may) then look like
$$\frac{U^{n+1} - U^n}{\Delta t} - \frac{1}{\Delta x^2} T_{\Delta x} U^p = 0.$$
p here depends on the method. if $p = n$ we have the EE, if $p = n + 1$ we have the IE(Remember simularity with test equation).
- Explain why the method of lines ODE can be considered stiff (frågan ställd för ekv. $u_t = u_{xx} + u_x$, med homogena Dirichletvillkor). "The problem is stiff because $T_{\Delta x} + S_{\Delta x}$ has large negative real eigen-values, hence strong exponential damping. This is due to $T_{\Delta x}$, i.e., it is due to the presence of the diffusion term u''_{xx} in the original equation. Because of the large negative eigenvalues, no matter what explicit time stepping method one would choose, there will be a CFL condition of the form $\frac{\Delta t}{\Delta x^2}$, which is prohibitive; the time stepping method “stalls,” and almost can't make any progress."

- Method of lines

$$u_t = u_{xx} + u_x \; homogenous\ BC,\;\; u(0,x) = g(x)))$$
$$u_{xx} = T_{\Delta x}\; u_x = S_{\Delta x}$$
$$A = T_{\Delta x} + S_{\Delta x}\; \dot{u} = Au$$
$$u^{n+1} = u^n + \frac{\Delta t}{2}(Au^{n+1} + Au^n)\; (discretization\ TR)$$
$$u^{n+1}(I - \frac{\Delta t A}{2}) = u^n(I + \frac{\Delta t A}{2})\; (linear\ system\ of\ equations\ TR)$$

- If the problem is stiff, use an implicit time stepping method ($1/T_{\Delta x}$ makes a problem stiff). This could be for example the trapezodial rule.
- Butcher tableau(get R(z) and determine stability):

0	0	0
1	1/3	2/3
b^T	1/3	2/3

$$hY_1' = hf(y_n)$$
$$hY_2' = hf(y_n + \frac{hY_1'}{3} + \frac{2hY_2'}{3})$$
$$y_{n+1} = y_n + \frac{hY_1'}{3} + \frac{2hY_2'}{3}$$
$$\dot{y} = \lambda y,\;\; y_1 = 1$$
$$hY_1' = h\lambda$$
$$hY_2' = h\lambda(1 + \frac{h\lambda}{3} + \frac{2hY_2'}{3})$$
$$hY_2'(1 - \frac{2h\lambda}{3}) = h\lambda(1 + \frac{h\lambda}{3})$$
$$R(h\lambda) = \frac{1+\frac{h\lambda}{3}}{1-\frac{2h\lambda}{3}}$$

Pole in RHS, check $R(hi\omega)$, less than 1 means A-stable

$$|R(hi\omega)|^2 = \left|\frac{1+\frac{h\lambda}{3}}{1-\frac{2h\lambda}{3}}\right| = \frac{1+\frac{h^2\lambda^2}{9}}{1+\frac{4h^2\lambda^2}{9}} \le 1$$

0	0	0	0
1/2	1/2	0	0
3/4	0	3/4	0
b^T	2/9	3/9	4/9

$$hY_1' = hf(y_n)$$
$$hY_2' = hf(y_n + \frac{hY_1'}{2})$$
$$hY_3' = hf(y_n + \frac{3hY_2'}{4})$$
$$y_{n+1} = y_n + \frac{2hY_1'}{9} + \frac{3hY_2'}{9} + \frac{4hY_3'}{9}$$
$$\dot{y} = \lambda y,\;\; y_1 = 1$$
$$hY_1' = h\lambda$$
$$hY_2' = h\lambda(1 + \frac{h\lambda}{2})$$
$$hY_3' = h\lambda(1 + \frac{3h\lambda}{4} + \frac{(3h\lambda)^2}{8})$$
$$P(h\lambda) = 1 + \frac{1}{9}(2h\lambda + 3h\lambda + \frac{3(h\lambda)^2}{2} + 4h\lambda + 3(h\lambda)^2 + \frac{3(h\lambda)^3}{2})$$
$$P(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6}$$

- Normuppgift $u_t + (f(u))_x = 0$, $u(t,0) = u(t,1)$, $||u(t,\cdot)||_2$ remains constant in time. $||u(t,\cdot)||_2^2 = \int_0^1 u \cdot u dx$, $u_t = -(f(u))_x$.

$$0 = \frac{d}{dt}||u(t,\cdot)||_2^2 = \frac{d}{dt}\int_0^1 u{\cdot}udx = \int_0^1 u_t{\cdot}udx = -\int_0^1 (f(u))_x{\cdot}udx = [partialintegrera] =$$

-[f(u)u]$_0^1$ $+ \int_0^1 f(u)u_x dx = [F(u)]_0^1 = 0$

I näst sista steget är den första termen noll eftersom u (och f:s) funktionsvärden är desamma i 0 och 1. Det sista steget kan tas eftersom u_x är den inre derivatan till f (alternativt gör ett variabelbyte med $\frac{du}{dx} = u_x$). $[F] = 0$ ty det är samma input i de två fallen. Alternativ lösning

$$< u, u' >= [uu]_0^1 - < u', u >= [uu]_0^1 - < u, u' > \implies$$
$$2 < u, u' >= u(1)^2 - u(0)^2 = 0 \implies < u, u' >= 0$$

$$< u, u_t >=< u, (f(u))_x >= - < u_x, f(u) >=$$
$$-\int_0^1 u' f(u)dx = -\int_{u(t,0)}^{u(t,1)} f(u)du = 0$$
$$\frac{d}{dt}\|u(t,\cdot)\|_2^2 = \frac{d}{dt} < u, u >= 2 < u, u_t >= 0$$

- Runge-Kutta, order of consistency

$$y_{n+1} = y_n + h(a_1 f(t_n, y_n) + a_2 f(t_n + b_1 h, y_n + b_2 h f(t_n, y_n)))$$
$$Order\ 1$$
$$|y(t_n) - \hat{y_n}| = 1y(t_n) + y'(t_n)h + O(h^2)$$
$$-(y(t_n) + ha_1 y'(t_n) + a_2 y'(t_n + b_1 h, y_n + b_2 h y'(t_n)))$$
$$\to a_1 = 1\; a_2 = 0\; b_1 = b_2 = 0 \implies a_1 + a_2 = 1$$
$$Order\ 2$$
$$|y(t_{n+1}) - \hat{y}_{n+1}| = 1y(t_n) + y'(t_n)h + \frac{y''(t_n)h^2}{2} + O_1(h^3)$$
$$-(y(t_n) + ha_1 y'(t_n) + ha_2(f(t_n, y_n) +$$
$$f_t'(t_n, y_n)b_1 h + f_y'(t_n, y_n)b_2 h f(t_n, y_n) + O_2(h^2)))$$
$$= hy'(1 - (a_1 + a_2) + h^2(\frac{y''(t_n)}{2} -$$
$$a_2 b_1 y''(t_n) + b_2 y'(t_n) f_y'(t_n, y_n) + O_2(h^3)) + O_1(h^3)$$
$$\to a_1 + a_2 = 1\; b_2 = 0\; a_2 b_1 = \frac{1}{2}$$

- Analytical eigenvalue problem

$$\mathcal{L}_\alpha u = \lambda u\; u(0) = u(1) = 0\; on\; [0,1]$$
$$\mathcal{L}_\alpha u = u'' + \alpha u' \to$$
$$u'' + \alpha u' = \lambda u \to kar.pol\; \kappa^2 + \alpha\kappa - \lambda = 0 \implies \kappa = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \lambda}$$
$$\gamma = \sqrt{\frac{\alpha^2}{4} + \lambda} \implies u(x) = e^{-\alpha x/2}(Ae^{\gamma x} + Be^{-\gamma x})$$
$$u(0) = 0 \implies A = -B,\;\; u(1) = 0 \implies \gamma = i\omega \to$$
$$u_k(x) = e^{-\alpha x/2} sin\omega_k x,\;\; \omega_k = k\pi \implies u(1) = 0 \to$$
$$u_k(x) = e^{-\alpha x/2} sink\pi x \implies -\omega_k^2 = -k^2\pi^2 = \lambda_k + \frac{\alpha^2}{4}$$
$$\implies \lambda_k = -(k\pi)^2 - \frac{\alpha^2}{4}$$

Discrete eigenvalue problem (samma operator, söker reella egenvärden)

$$Au = \lambda_{\Delta x} u\;,\;\; A = \frac{1}{\Delta x^2} Tridiag(1 - \alpha\Delta x/2 \quad -2 \quad 1 + \alpha\Delta x/2)$$
$$\lambda_k[A] = -\frac{2}{\Delta x^2} + \frac{2}{\Delta x^2}\sqrt{1 - \alpha^2\Delta x^2/4}\, cos\frac{k\pi}{N+1}$$
$$real\ iff\;\; |\alpha\Delta x| < 2 \implies \Delta x \le 2/|\alpha| \iff N + 1 \ge |\alpha|/2$$

Exam term 2 2018:

1: Implicit midpoint rule is:

$$y_{n+1} = y_n + \Delta t f(t_n + \Delta t/2, (y_n + y_{n+1})/2$$

Stability region:

$$y' = \lambda y \implies \Delta t/2, (y_n + y_{n+1})/2 = \Delta t\lambda(y_n + y_{n+1})/2 \implies \frac{y_{n+1}}{y_n} = \frac{1 + h/2}{1 - h\lambda/2}$$

h>0, Re(λ) $< 0 \implies |R(h\lambda)| < 1 \implies A - stable$

Butcher array:

0	0
1/2	0

2: system of ODEs

$$\dot{y} = (\begin{matrix} -6y_1y_2 & 4 \\ 4y_1 & -6y_2^2 \end{matrix})y$$

one step of explicit euler:

$$y_{n+1} = (\begin{matrix} y_1^n \\ y_2^n \end{matrix}) + h(\begin{matrix} -6y_1^n y_2^n & 4 \\ 4y_1^n & -6y_2^{n2} \end{matrix})(\begin{matrix} y_1^n \\ y_2^n \end{matrix})$$

one step of trapeziodal:

$$y_{n+1} = (\begin{matrix} y_1^n \\ y_2^n \end{matrix}) + h/2(\begin{matrix} -6y_1^n y_2^n & 4 \\ 4y_1^n & -6y_2^{n2} \end{matrix})(\begin{matrix} y_1^n \\ y_2^n \end{matrix}) + h/2(\begin{matrix} -6y_1^{n+1} y_2^{n+1} & 4 \\ 4y_1^{n+1} & -6y_2^{n+12} \end{matrix})(\begin{matrix} y_1^{n+1} \\ y_2^{n+1} \end{matrix})$$

Newton's method, a way to start the iteration, and a termination criterion:

$$y^{i+1} = y^i - F(y^i)F'^{-1}(y^i), F'jacobian(y' = \frac{y_{n+1} - y_{n-1}}{2\Delta x^2} \qquad (1)$$

Initial guess: explicit method, termination criterion: $||y^{i+1} - y^i|| < TOL$3: system of ODEs

$$\dot{y} = (\begin{matrix} -6 & 4 \\ 4 & -6 \end{matrix})y$$

largest time step expl euler: matrix eigenvalues ($\det(\lambda I - A)$) -> put in R(z) (y_{n+1}/y_n). Impl euler: A-stable -> no restriction. Multiply by 100 -> large negative eigenvalues -> stiff -> implicit euler.

4: butcher tableau

0	0	0
1	1	0
	1/2	1/2

perform one R-K step for $\dot{\theta} = -9.81 sin(\theta)$

eq-system: $x_1 = \theta \quad \dot{x}_1 = x_2$, $x_2 = \dot{\theta}\; \dot{x}_2 = -9.81(x_1)$ $Y_1 = y_n, Y_1' = f(t_n, y_n), Y_2 =$ $y_n + \Delta t f(t_n, y_n), Y_2' = f(t_n + \Delta t, y_n + \Delta t(f(t_n, y_n), y_{n+1} =$ $y_n + \Delta t/2(f(t_n, y_n) + f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n)) =$

$$\begin{bmatrix} x_1^n \\ x_2^n \end{bmatrix} + \Delta t/2(\begin{bmatrix} x_2^n \\ -9.81sin(x_1^n) \end{bmatrix} + \begin{bmatrix} x_2^n \\ -9.81sin(x_1^n) \end{bmatrix} + 2\Delta t\begin{bmatrix} x_2^n \\ -9.81sin(x_1^n) \end{bmatrix})$$