5 Komplex differentialkalkyl

5.2 $f = u + iv$, $z = x + iy$

a) $f(z) = (1+i)z = (1+i)(x+iy) = x + ix + iy - y =$

$$= (x-y) + i(x+y)$$

$$\boxed{u = x - y, \quad v = x + y}$$

b) $f(z) = z^3 = (x+iy)^3 = (x^2 + 2ixy - y^2)(x+iy) =$

$$= x^3 + 2ix^2y - y^2x + ix^2y - 2xy^2 - iy^3 =$$

$$= x^3 - y^2x - 2xy^2 + i(3x^2y - y^3)$$

$$\boxed{\begin{aligned} u &= x^3 - 3xy^2 \\ v &= 3x^2y - y^3 \end{aligned}}$$

c) $f(z) = e^{2z} = e^{2(x+iy)} = e^{2x} \cdot e^{2iy} =$

$$= e^{2x} \cdot (\cos 2y + i \sin 2y)$$

$$\boxed{u = e^{2x} \cos 2y, \quad v = e^{2x} \sin 2x}$$

d) $f(z) = \bar{z} = x - iy \Rightarrow \boxed{u = x \,,\ v = -y}$

e)
$$f(z) = \frac{1}{z - i} = \frac{1}{x + iy - i} = \frac{1}{x + i(y-1)} = \frac{1 \cdot \bar{p}}{p \cdot \bar{p}} =$$

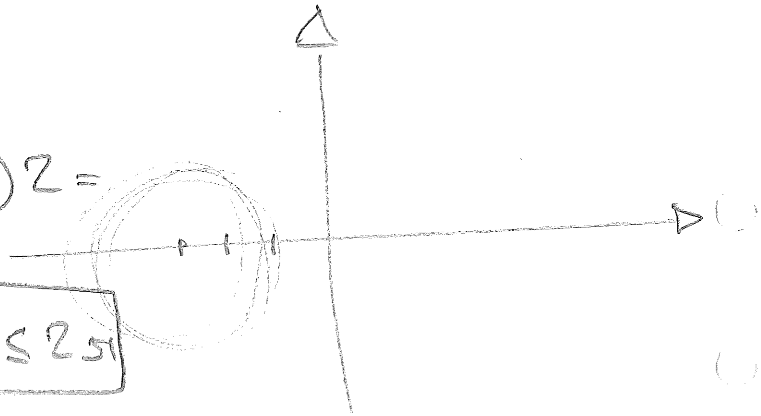
$$= \frac{x - i(y-1)}{x^2 + (y-1)^2} = \frac{x}{x^2+(y-1)^2} + i\,\frac{-(y-1)}{x^2+(y-1)^2}$$

$$\boxed{u = \frac{x}{x^2+(y-1)^2} \,,\quad v = -\frac{y-1}{x^2+(y-1)^2}}$$

5.4 a) $|z+3| = 2$

$$z = -3 + (i\sin\theta + \cos\theta)2 =$$

$$= \boxed{-3 + 2e^{i\theta} \,,\quad 0 \le \theta \le 2\pi}$$



b) $\boxed{-3 + 2e^{-i\theta} \,,\quad 0 \le \theta \le 6\pi}$

5.6

a) $f(z) = Re(z) = a + 0 \cdot i$

$f(z)$ är ej ~~de~~ komplext deriverbar någonstans, ej analytisk på ngt öppet område

b) $f(z) = |z|^2 = \left(\sqrt{a^2+b^2}\right)^2 = a^2 + b^2$

I origo, nej $\boxed{VF?}$

c) $f(z) = z^2$, överallt, ja.

d) $f(z) = z^{-2} = \frac{1}{z^2}$, överallt utom origo
Ja - " ——

e) $\bar{z}^2 = (a-bi)^2$, $\boxed{\text{I origo, nej}}$ VF?

5.10 $f = u + iv$ är analytiskt i ett öppet intervall.

$Im(f) = \text{konstant} \Leftrightarrow f = \text{konstant}$

$Im(f) = v$, $v = \text{konst} \Rightarrow v'_x = v'_y = 0$

Cauchy Riemans ekvationer $\Rightarrow u'_x = u'_y = 0$

så $f = \text{konstant}$ #

$$\begin{cases} u'_x = v'_y \\ u'_y = -v'_x \end{cases}$$

$\boxed{5.12}$

a) $f = e^x \cdot \sin(y)$

$f'_x = e^x \sin(y)$ $\quad$ $f'_y = e^x \cos y$

$f''_{xx} = e^x \cdot \sin(y)$ $\quad$ $f''_{yy} = -e^x \sin(y)$

$f''_{xx} + f''_{yy} = 0 \Rightarrow \boxed{\text{Harmonisk}}$

b) $f = x^2 + y^2$

$f'_x = 2x$ $\quad$ $f'_y = 2y$

$f''_{xx} = 2$ $\quad$ $f''_{yy} = 2$

$f''_{xx} + f''_{yy} = 2 + 2 \neq 0 \Rightarrow \boxed{\text{EJ harmonisk}}$

c) $f = x^2 - y^2$

$f''_{xx} = 2$, $f''_{yy} = -2$

$f''_{xx} + f''_{yy} = 2 - 2 = 0 \Rightarrow \boxed{\text{Harmonisk}}$

$\boxed{5.14}$

$$\boxed{U(x,y) = x^3 - 3xy^2 + 2x - 1}$$

$U'_x = 3x^2 - 3y^2 + 2$ | $U'_y = -6xy$

$U''_{xx} = 3 \cdot 2x = 6x$ | $U''_{yy} = -6x$

$$U''_{xx} + U''_{yy} = 6x - 6x = 0 \Rightarrow \boxed{\text{Harmonisk}}$$

$$\begin{cases} U'_x = V'_y \\ U'_y = -V'_x \end{cases} \Rightarrow \begin{array}{l} V'_y = 3x^2 - 3y^2 + 2 \\ +V'_x = +6xy \end{array}$$

$V = 3x^2y + g(y)$

$V'_y = 3x^2 + g'(y) = 3x^2 - 3y^2 + 2$

$\Rightarrow g'(y) = -3y^2 + 2$

$g(y) = -y^3 + 2y + C$

$$\Rightarrow \boxed{V(x,y) = 3x^2y - y^3 + 2y + C}$$

$\boxed{5.15}$

$$u(x,y) = y^3 + ax^2y = \mathrm{Re}(f(z))$$

$$u'_x = 2axy \qquad u'_y = 3y^2 + ax^2$$

$$u''_{xx} = 2ay \qquad u''_{yy} = 6y$$

$$v'_y = 2axy \qquad -v'_x = 3y^2 + ax^2$$

$$v = axy^2 + g(x)$$

$$v'_x = ay^2 + g'(x) = -3y^2 - ax^2$$

$$\Rightarrow \boxed{a = -3} \qquad g(x) = x^3 + C$$

$$f(z) = y^3 - 3x^2y + (-3xy^2 + x^3 + C)\,i =$$

$$= y^3 - 3x^2y - i3xy^2 + ix^3 + iC =$$

$$= i(x+iy)^3 + iC = iz + iC =$$

$$= \boxed{i(z+C)}$$