

Föreläsning 2

Repetition: $x_n - rx_{n-1} = 0$ har lösning:

$$x_n = C \cdot r^n$$

Vi studerar en ekvation av grad ℓ :

$$x_n + ax_{n-1} + bx_{n-2} = 0$$

$$\text{Ansats: } Cr^n + Car^{n-1} + br^{n-2} = 0$$

$$Cr^{n-2}(r^2 + ar + b) = 0$$

$p(r)$ karakteristiskt polynom.

$$\text{Om } p(r_1) = p(r_2) = 0$$

$$\Rightarrow r^2 + ar + b = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + \underline{\underline{r_1 r_2}}$$

Två relationer:

$$\left\{ \begin{array}{l} a = -(r_1 + r_2) \\ b = r_1 r_2 \end{array} \right.$$

SATS

$x_n + ax_{n-1} + bx_{n-2} = 0$ har lösningar

$$x_n = \begin{cases} A\gamma^n + B\gamma_2^n & \text{om } \gamma_1 \neq \gamma_2 \\ A\gamma_1^n + Bn\gamma_2^n & \text{om } \gamma_1 = \gamma_2 \end{cases}$$

$$= (A + Bn)\gamma_1^n$$

Bevis

Låt $y_n = x_n - \gamma_1 x_{n-1}$

$$y_n - \gamma_2 y_{n-1} = (x_n - \gamma_1 x_{n-1}) - \gamma_2 (x_{n-1} - \gamma_1 x_{n-2}) =$$

$$= x_n - (\gamma_1 + \gamma_2) x_{n-1} + \gamma_1 \gamma_2 x_{n-2} =$$

$$= x_n + (\alpha)x_{n-1} + (\beta)x_{n-2} = 0$$

Detta ger:

$$y_n - \gamma_2 y_{n-1} = 0 \Rightarrow \boxed{y_n = C \cdot \gamma_2^n}$$

Vad betyder det för x_n ?

$$\text{Jö, } \Rightarrow \boxed{x_n - \gamma_1 x_{n-1} = C \gamma_2^n = y_n}$$

Lös homogena \Leftrightarrow Partikulär,

$$X_n^h - r_1 X_{n-1}^h = 0 \Rightarrow X_n^h = A r_1^n \quad \text{homogen!}$$

$$X_n^p = \begin{cases} B r_2^n & \text{om } r_1 \neq r_2 \\ B n r_2^n & \text{om } r_1 = r_2 \end{cases}$$

Sö

$$X_n = X_n^h + X_n^p = \begin{cases} A r_1^n + B r_2^n & \text{om } r_1 \neq r_2 \\ A r_1^n + B n r_1^n & \text{om } r_1 = r_2 \end{cases}$$

$\epsilon \in \mathbb{PZ}$

Exempel:

$$\begin{cases} X_n - 5X_{n-1} + 6X_{n-2} = 4^n \\ X_0 = 1, X_1 = 2 \end{cases}$$

$$D(r) = r^2 - 5r + 6 = 0$$

$$\Rightarrow r_1 = 2, r_2 = 3$$

$$\therefore \Rightarrow X_n^h = A \cdot 2^n + B \cdot 3^n$$

$$\begin{aligned} & \triangleright X_n^p = k \cdot 4^n \\ & * \text{ stoppar in *} \\ & k \cdot 4^n = 5 \cdot k \cdot 4^{n-1} + 6 \cdot k \cdot 4^{n-2} \cdot 4^n \\ & \Leftrightarrow 16k - 20k + 6k = 16 \\ & \Leftrightarrow k = 8 \end{aligned}$$

$$X_n = X_n^h + X_n^p = A \cdot 2^n + B \cdot 3^n + 8 \cdot 4^n$$

Insättning ger $A \neq B$:

$$\therefore \begin{array}{l} A = 9 \\ B = -16 \end{array}$$

$$\Rightarrow X_n = 9 \cdot 2^n - 16 \cdot 3^n + 8 \cdot 4^n$$

Exempel

$$X_n + 2X_{n-1} + 2X_{n-2} = 0$$

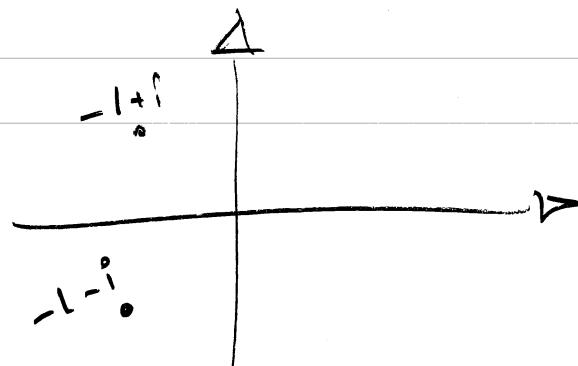
$$p(r) = r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i$$

$$\rightarrow X_n = A(-1+i)^n + B(-1-i)^n$$

ser komplex ut fast det ej är det.

Reell form:

$$\left\{ \begin{array}{l} -1+i = \sqrt{2} e^{i\frac{3\pi}{4}} \\ -1-i = \sqrt{2} e^{-i\frac{3\pi}{4}} \end{array} \right.$$



$$X_n = A\sqrt{2}^n e^{i\frac{3\pi}{4}n} + B\sqrt{2}^n e^{-i\frac{3\pi}{4}n}$$

EULER SÄGER

$$X_n = \sqrt{2}^n \left(A \left(\cos \frac{3\pi n}{4} + i \sin \frac{3\pi n}{4} \right) + B \left(\cos \frac{3\pi n}{4} - i \sin \frac{3\pi n}{4} \right) \right)$$

$$= \sqrt{2}^n \left(\cos \frac{3\pi n}{4} \cdot \underbrace{(A+B)}_C + \sin \frac{3\pi n}{4} \underbrace{(iA - iB)}_D \right) =$$

$$= \boxed{C \sqrt{2}^n \cos \left(\frac{3\pi n}{4} \right) + D \sqrt{2}^n \sin \left(\frac{3\pi n}{4} \right)}$$

Exempel

$$X_n + aX_{n-1} + bX_{n-2} = 0, n \geq 2$$

$$X_0 = 0, X_1 = 1, X_2 = 4, X_3 = 37$$

Stoppar i n

$$X_2 + aX_1 + bX_0 = 0$$

$$4 + a + 0 = 0 \Rightarrow \boxed{a = -4}$$

$$X_3 + aX_2 + bX_1 = 0$$

$$37 + 4a + b = 0$$

$$\Rightarrow b = 37 - 4(-4) = \boxed{-21}$$

$$\Rightarrow \boxed{X_n - 4X_{n-1} - 21X_{n-2} = 0} \leftarrow \text{Enkel att l\"osa}$$

$$p(r) = r^2 - 4r - 21 = 0 \Rightarrow r_1 = 7, r_2 = -3$$

$$X_n = A \cdot 7^n + B \cdot (-3)^n$$

*NP använder begynnelsevillkoren r_1, r_2 *

$$\Rightarrow A = \frac{1}{10}, B = -\frac{1}{10}$$

$$\boxed{\text{Svar: } \frac{1}{10}(7^n - (-3)^n) = X_n}$$

Exempel

$$x_{n+2} - x_n = \sin\left(n \cdot \frac{\pi}{2}\right), n \geq 0$$

$$x_0 = 1, x_1 = 1$$

bla, bla, bla ... osv

$$P(r) = r^2 - 1 = 0, r = \pm 1$$

$$x_n^h = C \cdot 1^n + D(-1)^n$$

Skriv om sinus

$$\sin\left(\frac{\pi}{2} \cdot n\right) = \frac{e^{i\frac{\pi}{2}n} - e^{-i\frac{\pi}{2}n}}{2i} = \frac{i^n - (-i)^n}{2i}$$

$$x_n^p = A \cdot i^n + B(-i)^n$$

$$(A i^{n+2} + B(-i)^{n+2}) - (A i^n + B(-i)^n) = \frac{i^n - (-i)^n}{2i}$$

$$A i^{n+2} - A i^n = \frac{i^n}{2}$$

$$B(-i)^{n+2} - B(-i)^n = \frac{-(-i)^n}{2i}$$

$$\begin{cases} A^{i^2} - A = \frac{1}{2i} \\ B(-i)^2 - B = -\frac{1}{2i} \end{cases} \Leftrightarrow \begin{cases} -2A = \frac{1}{2i} \\ -2B = -\frac{1}{2i} \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{4i} = \frac{1}{4} \\ B = \frac{1}{4i} = -\frac{i}{4} \end{cases}$$

So $A = \frac{1}{4}, B = -\frac{i}{4}$

⇒ $X_n = X_n^h + X_n^p = C + D(-1)^n + \frac{1}{4}(i)^n - \frac{i}{4}(-i)^n$

* sen Gauß *

$$\begin{cases} X_0 = C + D + \frac{1}{4} - \frac{i}{4} = 1 \\ X_1 = D - D + \frac{i^2}{4} + \frac{i^2}{4} = 1 \end{cases} \stackrel{(\Rightarrow)}{\quad} \begin{cases} C + D = 1 \\ C - D = \frac{3}{2} \end{cases}$$

⇒ $\begin{cases} C = \frac{5}{4} \\ D = -\frac{1}{4} \end{cases}$

$$X_n = \frac{5}{4} - \frac{1}{4}(-1)^n + \frac{i^{n+1}}{4} + \frac{(-i)^{n+1}}{4} =$$

= $\boxed{\frac{5}{4} - \frac{1}{4}(-1)^n - \frac{1}{2} \sin(n \cdot \frac{\pi}{2})}$

Exempel

$$\rightarrow X_{n+2} - X_{n+1} - 2X_n = 2^n, \quad \begin{cases} n \geq 0 \\ X_0 = 0 \\ X_1 = 0 \end{cases}$$

$$D(r) = r^2 - r - 2 = 0$$

$$\Rightarrow \boxed{r_1 = -1, r_2 = 2}$$

$$X_n^h = C_1(-1)^n + D(2)^n$$

$$X_n^p = B \cdot n \cdot 2^n \quad \text{eftersom } 2 \text{ är en rot!}$$

*sätt in X_n^p *

$$B(n+2)2^{n+2} - B(n+1)2^{n+1} - 2Bn2^n = 2^n$$

$$\Leftrightarrow B(n+2) \cdot 4 - B(n+1) \cdot 2 - 2Bn = 1$$

$$\Leftrightarrow (4B - 2B - 2B)n + 8B - 2B = 1$$

$$\Leftrightarrow \boxed{B = \frac{1}{8}}$$

OSU, OSU OSU \Rightarrow

Gauss
Gauss
Gauss

$$\boxed{X_n = \frac{1}{8}(-1)^n + \frac{1}{8} \cdot 2^n + \frac{n \cdot 2^n}{8}}$$