Sammanfattning Numdiff.

Some PDE:s.

Equation	Name	CFL	Type
$\underbrace{u_t - u_{xx}}_{\text{diffusion}} = \underbrace{f(x)}_{\text{reaction}}$	Diffusion/reaction	$\frac{\Delta t}{\Delta x^2} \leq C$	Parabolic
$u_t + a u_x = 0$	Advection	$\frac{\Delta t}{\Delta x} \leq C$	Hyperbolic
$u_t - u_x - u_{xx} = 0$	Convection/Diffusion.	$\frac{\Delta t}{\Delta x^2} \leq C$	Parabolic
$i \cdot u_t = u_{xx}$	Schrödinger	$\frac{\Delta t}{\Delta x} \leq C$	Hyperbolic
$\begin{cases} u_t - u u_x = u_{xx} \\ u_t - \frac{1}{2}(u^2)_x = u_{xx} \end{cases}$	(Viscous) Burgers'	$\frac{\Delta t}{\Delta x^2} \leq C$	Parabolic
$u_t - u u_x = 0$	Inviscid Burgers'	$\frac{\Delta t}{\Delta x} \leq C$	Hyperbolic
$u_{tt} - u_{xx} = 0$	Wave	$\frac{\Delta t}{\Delta x} \leq C$	Hyperbolic
$u_t + u u_x = -u_{xxx}$	Korteweg-de Wries	$\frac{\Delta t}{\Delta x^3} \leq C$	Hyperbolic
$u_{xx} + u_{yy} = 0$	Laplace	—	Elliptic
$u_{xx} + u_{yy} = f(x)$	Poisson	—	Elliptic

For CFL-condition $\frac{\Delta t}{\Delta x}$ we only need to use an explicit method. Otherwise use an implicit method.

To check wether or not an equation is hyperbolic or parabolic, compare the highest order of the t and x derivatives.
If the difference is even it's hyperbolic, i.e. $u_{tt} = u_{xx}$ or $u_t = u_x$
—— " —— uneven it's parabolic, i.e. $u_t = u_{xx}$ or $u_t = u_{xxx}$

Initial value problems

The backward difference operator is defined as

$$\nabla y_n = y_n - y_{n-1}$$

$$\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$$

You solve problems of the form

$$y' = f(y), \quad y(0) = y_0$$

with methods that look like (where $\Delta t = h$)

$$\underbrace{\nabla y_n}_{} = h \cdot \underbrace{f(y_n)}_{} \qquad (*)$$

Any combination of dif. operators

To find the order of consistency you set

$$\begin{cases} y = P(t) \\ f = P'(t) \end{cases} \quad \text{for } P(t) = t^m \text{ where } t_n = 2h,\ t_{n-1} = h,\ t_{n-2} = 0$$

$$(\text{or } t_{n+1} = h,\ t_n = 0,\ t_{n-1} = -h)$$

Then you insert this into $(*)$ for $m = 0, 1, \ldots$ If the LHS = RHS the method is of least that order.

To check zero-stability, investigate $y' = 0$ and compute the characte[r] equation. Put the "lowest" order of y_n (i.e. y_{n-1} if you have y_{n+1}, y_n and y_{n-1}) as z^0 and contiue upwards with increasing powers of z. Example

$$y_{n+1} + 2y_n + 3y_{n-1} + 4y_{n-2} = 0 \quad \Rightarrow \quad z^3 + 2z^2 + 3z + 4 = 0.$$

Find the roots. If the roots are simple and $|z| < 1$ the root condition is fulfilled and the method is zero-stable

To compute the next step for an implicit method, collect all known values on the RHS and the others on the left

$$y_n + hf(y_n) = \underbrace{y_{n-1} + y_{n-2}}_{\text{can be summarized as a function of}}.$$

Solve using Newton's method.

Boundary value problems

Runge-Kutta

We have a Butcher tableau:

t=0	0	0	0	0
t=1	1/2	1/2	0	0
t=2	1	1/4	3/4	0
	y	α	β	γ
		y=1	y=2	y=3

If the diagonale is zero and lower triangular it's explicit.

From this you can formulate the scheme

$$\begin{cases} Y_1' = f(t_n, y_n) \\ Y_2' = f(t_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h \cdot Y_1') \\ Y_3' = f(t_n + h,\ y_n + \tfrac{1}{4}hY_1' + \tfrac{3}{4}h \cdot Y_2') \end{cases} \Leftrightarrow \begin{cases} Y_1' = y_n \\ Y_2' = y_n + \tfrac{1}{2}hY_1' \\ Y_3' = y_n + \tfrac{1}{4}hY_1' + \tfrac{3}{4}hY_2' \end{cases}$$

$$y_{n+1} = y_n + \alpha h \cdot Y_1' + \beta h Y_2' + \gamma \cdot h \cdot Y_3'$$

To find the stability function $R(\lambda h)$, investigate the linear test function $y' = \lambda y$.

We then set

$$\varphi_1' = \lambda y_n$$

$$\varphi_2' = \lambda \cdot (y_n + \tfrac{1}{2} h \varphi_1') = \lambda (1 + \tfrac{1}{2} h \lambda) y_n$$

Then $R(\lambda h)$ is y_{n+1} for $y_n = 1$.

To check A-stability we check if

$$|R(\lambda h)| \leq 1 \quad \text{for large steps } h.$$

One can also check if the method is explicit because if it is, then it's not A-stable!

To check if it's of a certain order, compare it to a Taylor series expansion

$$\underbrace{y_n}_{\text{order }0} + \underbrace{\frac{y_n'}{1!}}_{\text{order }1} + \underbrace{\frac{y''}{2!}}_{\text{order }2} + \underbrace{\frac{y_n'''}{3!}}_{\text{order }3} + \dots$$

Discretization of 2p BVP:s

Use these discretizations:

$$y_n' = \frac{y_{n+1} - y_{n-1}}{2\Delta x}$$

$$y_n'' = \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2}$$

For a problem

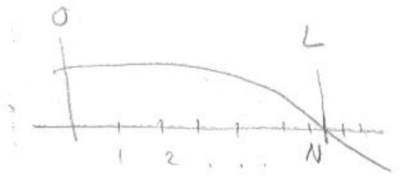
$$u'' - u' = f(x), \quad 0 \leq x \leq L, \quad t > 0$$

Some BV

we first set up a grid.

If we have Neumann condition at $x=L$ we do as following

$u'(L)=0$



$\Delta x = \frac{L}{N+1/2}$, N interior points

with $x_i = i \cdot \Delta x$ for $i=1:N$.

Then we can approximate $u'(L)=0$ with

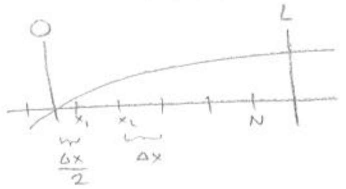
$$\frac{u_{N+1}-u_N}{\Delta x} = 0 \iff u_{N+1} = u_N$$

since

$$\frac{x_{N+1}+x_N}{2} = \frac{(N+1)\Delta x + N\Delta x}{2} = \frac{(N+1)\cdot\frac{L}{N+\frac{1}{2}} + N\cdot\frac{L}{N+1/2}}{2}$$

$$= \frac{L}{2}\left(\frac{2N+1}{N+1/2}\right) = L$$

If we instead have $u'(0)=0$:



$\Delta x = \frac{L}{N+1/2}$, N interior points

with $x_i = (i-\frac{1}{2})\Delta x$ for $i=1:N$.

Then we approximate $u'(0)=0$ with

$$\frac{u_0+u_1}{\Delta x} = 0 \iff u_1 = u_0$$

since

$$\frac{x_0+x_1}{2} = \frac{(0-\frac{1}{2})\Delta x + (1-\frac{1}{2})\Delta x}{2} = 0$$

Now compute the finite symmetric differances.

$$\frac{u_0 - 2u_1 + u_2}{\Delta x^2} - \frac{-u_0+u_2}{2\Delta x} = f(x_1)$$

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} - \frac{-u_{i-1}+u_{i+1}}{2\Delta x} = f(x_i), \quad i=2:N-1$$

$$\frac{u_{N-1} - 2u_N + u_{N+1}}{\Delta x^2} - \frac{-u_{N-1}+u_{N+1}}{2\Delta x} = f(x_N)$$

Replace values as needed! (u_0 or u_{N+1})

Lax-Freidrichs scheme

We have an equation of some kind with $u(x,0) = \phi_0(x)$ for $0 < x \leq 1$, $t > 0$. For example $u_t + a u_x = 0$.

The scheme looks like

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) + \frac{a \Delta t}{2 \Delta x}(u_{j+1}^n - u_{j-1}^n) \quad , \quad \mu = \frac{\Delta t}{\Delta x}$$

$$= \frac{1}{2}\left((1 + a\mu) u_{j+1}^n + (1 - a\mu) u_{j-1}^n \right)$$

If we have periodic boundary values and set $a^+ = 1 + a\mu$ and $a^- = 1 - a\mu$ we get the matrix.

$$T_{\Delta x} = \begin{bmatrix} 0 & a^- & \cdots & & a^+ \\ a^+ & 0 & a^- & & 0 \\ 0 & a^+ & 0 & a^- & \cdots \\ & & & \ddots & \\ a^- & \cdots & & a^+ & 0 \end{bmatrix} , \qquad U^0 = \begin{bmatrix} \phi_0(x_1) \\ \phi_0(x_2) \\ \vdots \\ \phi_0(x_N) \end{bmatrix}$$

We iterate over

$$U^{n+1} = T_{\Delta x} \cdot U^n$$

starting from U^0 to get the solution.

We can categorize $T_{\Delta x}$: (it can have many categories)

Toeplitz: $T_{\Delta x} = \begin{bmatrix} a & b & 0 & 0 \\ c & a & b & 0 \\ 0 & c & a & b \\ 0 & 0 & c & a \end{bmatrix}$, Three diagonales, can be circulant tho. which is non-zero.

Symmetric: $T_{\Delta x} = T_{\Delta x}^T$

Skew-symmetric: $T_{\Delta x} = -T_{\Delta x}^T$

Circulant: $T_{\Delta x} = \begin{bmatrix} a & b & 0 & c \\ c & a & b & 0 \\ 0 & c & a & b \\ b & 0 & c & a \end{bmatrix}$.

Lax-Freidrich continues

If $a\mu = 1$ we get

$$T_{1/a} = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & 1 & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}$$

which is a cyclic permution matrix with $T_{1/a} = I$.
After N steps the wave is back at its original position.

If $T_{1/a}$ has eigenvalues λ then $T_{1/a}^N$ has λ^N. But since

$T_{1/a}^N = I \Rightarrow \lambda^N = 1$.

We then have

$$\lambda_k = e^{2\pi i k/N}$$

Because all eigenvalues have unit modulus $(|\lambda| = 1)$ and are simple, it is stable!