Exercises Week 5

I. PROBLEM 4.2

(a)

$$\begin{aligned}\mathcal{T}(\mathbf{d})\mathcal{T}(\mathbf{d}')\left|\psi\right\rangle &= \int d^3r \mathcal{T}(\mathbf{d})\mathcal{T}(\mathbf{d}')\left|\mathbf{r}\right\rangle\left\langle\mathbf{r}|\psi\right\rangle \\ &= \int d^3r \left|\mathbf{r}+\mathbf{d}'+\mathbf{d}\right\rangle\left\langle\mathbf{r}|\psi\right\rangle ,\end{aligned}$$

$$\begin{aligned}\mathcal{T}(\mathbf{d}')\mathcal{T}(\mathbf{d})\left|\psi\right\rangle &= \int d^3r \mathcal{T}(\mathbf{d}')\mathcal{T}(\mathbf{d})\left|\mathbf{r}\right\rangle\left\langle\mathbf{r}|\psi\right\rangle \\ &= \int d^3r \left|\mathbf{r}+\mathbf{d}'+\mathbf{d}\right\rangle\left\langle\mathbf{r}|\psi\right\rangle ,\end{aligned}$$

This implies

$$[\mathcal{T}(\mathbf{d}),\mathcal{T}(\mathbf{d}')] = 0.$$

(b) $\mathcal{D}(\hat{\mathbf{n}},\phi)$ and $\mathcal{D}(\hat{\mathbf{n}}',\phi')$ do not commute in general. For example, a rotation about the x-axis followed by a rotation about the y-axis gives a different result when the order of the rotations is reversed.

(c) $\mathcal{T}(\mathbf{d})$ and π do not commute:

$$\begin{aligned}\mathcal{T}(\mathbf{d})\pi\left|\mathbf{r}\right\rangle &= \mathcal{T}(\mathbf{d})\left|-\mathbf{r}\right\rangle = \left|-\mathbf{r}+\mathbf{d}\right\rangle , \\ \pi\mathcal{T}(\mathbf{d})\left|\mathbf{r}\right\rangle &= \pi\left|\mathbf{r}+\mathbf{d}\right\rangle = \left|-\mathbf{r}-\mathbf{d}\right\rangle .\end{aligned}$$

(d) $\mathcal{D}(\hat{\mathbf{n}},\phi)$ and π commute:

$$\begin{aligned}\pi\mathcal{D}(R)\psi(\mathbf{r}) &= \pi\psi(R^{-1}\mathbf{r}) = \psi(-R^{-1}\mathbf{r}), \\ \mathcal{D}(R)\pi\psi(\mathbf{r}) &= \mathcal{D}(R)\psi(-\mathbf{r}) = \psi(-R^{-1}\mathbf{r}).\end{aligned}$$

II. PROBLEM 4.3

$$(AB + BA)\psi = (ab + ba)\psi = 0.$$

This means that either a or b is zero or both are zero. The momentum and parity operator anticommute so that momentum eigenstate is usually not parity eigenstate. The only possibility for having a common eigenstate is when the momentum is zero.

III. PROBLEM 4.4

(a) The spin-angular function is given in (3.8.64) p229 (Sakurai)

$$\begin{aligned} y_l^{j=l\pm\frac{1}{2},m} &= \pm\sqrt{\frac{l \pm m + 1/2}{2l+1}} Y_l^{m-\frac{1}{2}}(\theta,\varphi)\chi_+ \\ &+ \sqrt{\frac{l \mp m + 1/2}{2l+1}} Y_l^{m+\frac{1}{2}}(\theta,\varphi)\chi_- \\ &= \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \pm\sqrt{l \pm m + 1/2}Y_l^{m-\frac{1}{2}}(\theta,\varphi) \\ \sqrt{l \mp m + 1/2}Y_l^{m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix}. \end{aligned} \tag{1}$$

It is evident that it is an eigenfunction of $\mathbf{L}^2, \mathbf{S}^2, \mathbf{J}^2$, and J_z. Here, $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

For $l = 0, j = 1/2, m = 1/2$, we have

$$y_0^{j=\frac{1}{2},m=1/2} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b)

$$\begin{aligned} (\sigma \cdot \mathbf{x}) y_0^{j=\frac{1}{2},m=1/2} &= (x\sigma_x + y\sigma_y + z\sigma_z) y_0^{j=\frac{1}{2},m=1/2} \\ &= \frac{1}{\sqrt{4\pi}} \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{4\pi}} \begin{pmatrix} z \\ x + iy \end{pmatrix}. \end{aligned}$$

Using

$$x/r = \sin\theta\cos\varphi, \quad y/r = \sin\theta\sin\varphi, \;\; z/r = \cos\theta,$$

we find

$$(\sigma\cdot\mathbf{x})y_0^{j=\frac{1}{2},m=1/2} = \frac{r}{\sqrt{4\pi}}\begin{pmatrix}\cos\theta\\ \sin\theta e^{i\varphi}\end{pmatrix} = -r\begin{pmatrix}-\frac{1}{\sqrt{3}}Y_{1,0}\\ \sqrt{\frac{2}{3}}Y_{1,1}\end{pmatrix}$$

Comparison with the spin function gives

$$m = \frac{1}{2}, \;\; l = 1.$$

Taking the lower sign in (1) gives

$$(\sigma\cdot\mathbf{x})y_0^{j=\frac{1}{2},m=1/2} = -\frac{r}{\sqrt{3}}\begin{pmatrix}-\sqrt{1-1/2+1/2}Y_{1,0}\\ \sqrt{1+1/2+1/2}Y_{1,1}\end{pmatrix} = -ry_{l=1}^{j=\frac{1}{2},m=\frac{1}{2}}.$$

The effect of $(\sigma\cdot\mathbf{x})$ is to change $l = 0 \to 1$ keeping j and m unchanged.

(c) $(\sigma\cdot\mathbf{x})$ is a (pseudo) scalar or a spherical tensor of rank zero so that it cannot change the total j and m. Under space inversion $(\sigma\cdot\mathbf{x})$ is odd so it connects even $(l = 0)$ and odd parity $(l = 1)$.

IV. PROBLEM 4.5

A parity-violating potential between the atomic electron and the nucleus is given by

$$V = \lambda[\delta(\mathbf{x})\mathbf{S}\cdot\mathbf{p} + \mathbf{S}\cdot\mathbf{p}\delta(\mathbf{x})]$$

where $\mathbf{S}$ and $\mathbf{p}$ are the spin and momentum operators of the electron. This potential is a (pseudo) scalar, which, as in the Problem 4, connects even and odd l but cannot change j and m.

From first-order perturbation theory

$$C_{n'l'j'm'} = \frac{\langle n', l', j', m'|V|n, l, j, m\rangle}{E_{nlj} - E_{n'l'j'}}$$

where $l' = l \pm 1, \;\; m' = m, \;\; j' = j$. The wavefunction for $|n, l, j, m\rangle$ can be written as

$$\langle \mathbf{r}|n,l,j,m\rangle = R_{nlj}(r)y_l^{j=l\pm\frac{1}{2},m}(\theta,\varphi).$$

$$\begin{aligned}&\langle n',l',j',m'|V|n,l,j,m\rangle\\&=\lambda\int d^3r R_{n',l',j'}(r)y_{l'}^{j'=l'\pm\frac{1}{2},m'}(\theta,\varphi)[\delta(\mathbf{x})\mathbf{S}\cdot\mathbf{p}+\mathbf{S}\cdot\mathbf{p}\delta(\mathbf{x})]R_{nlj}(r)y_l^{j=l\pm\frac{1}{2},m}(\theta,\varphi).\end{aligned}$$

Evidently the matrix element vanishes unless R_{nlj} and $R_{n',l',j'}$ are finite at the origin.

V. PROBLEM 4.7

(a)

$$\psi(\mathbf{x},t)=\exp\left[i(\mathbf{k}\cdot\mathbf{x}-\omega t\right].$$

$$\begin{aligned}\psi^*(\mathbf{x},-t)&=\exp\left[-i(\mathbf{k}\cdot\mathbf{x}+\omega t\right]\\&=\exp\left[i(-\mathbf{k}\cdot\mathbf{x}-\omega t\right],\end{aligned}$$

i.e., the momentum direction is reversed.

(b) The eigenstate of $\sigma\cdot\hat{\mathbf{n}}$ with eigenvalue +1 is, from (3.2.52) p172 (Sakurai)

$$\chi_+(\hat{\mathbf{n}})=\begin{pmatrix}\cos\frac{\beta}{2}e^{-i\alpha/2}\\ \sin\frac{\beta}{2}e^{i\alpha/2}\end{pmatrix}$$

where α and β are the azimuthal and polar angle characterising $\hat{\mathbf{n}}$.

$$-i\sigma_y\chi_+^*(\hat{\mathbf{n}})=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2}e^{i\alpha/2}\\ \sin\frac{\beta}{2}e^{-i\alpha/2}\end{pmatrix}=\begin{pmatrix}-\sin\frac{\beta}{2}e^{-i\alpha/2}\\ \cos\frac{\beta}{2}e^{i\alpha/2}\end{pmatrix}.$$

This corresponds state can be obtained by

$$\begin{aligned}\alpha&\to\alpha+\pi,\\ \beta&\to\pi-\beta,\end{aligned}$$

in $\chi_+(\hat{\mathbf{n}})$, which rotates the spin up state $\chi_+(\hat{\mathbf{n}})$ into a spin down state.

VI. PROBLEM 4.8

(a) The proof can be found in the textbook.

(b) Because $\exp(-i\mathbf{p}\cdot\mathbf{x})$ is degenerate to $\exp(i\mathbf{p}\cdot\mathbf{x})$.

VII. PROBLEM 4.9

$$\begin{aligned}
\theta\left|\alpha\right\rangle &= \int d^3p\theta\left|p\right\rangle\left\langle p|\alpha\right\rangle \\
&= \int d^3p\left\langle p|\alpha\right\rangle^*\theta\left|p\right\rangle \\
&= \int d^3p\left\langle p|\alpha\right\rangle^*\left|-p\right\rangle \\
&= \int d^3p\left\langle -p|\alpha\right\rangle^*\left|p\right\rangle .
\end{aligned}$$

Therefore

$$\langle p|\theta|\alpha\rangle = \langle -p|\alpha\rangle^* = \phi^*(-p)$$

VIII. PROBLEM 4.12

The Hamiltonian of a spin one system is

$$H = AS_z^2 + B(S_x^2 - S_y^2).$$

From (3.5.41 p196 Sakurai) we can construct

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The Hamiltonian is then

$$H = \hbar^2\begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}.$$

The block matrix that need to be diagonalised is

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The eigenvalues are

$$E = \hbar^2(A \pm B) \text{ and } 0$$

and the corresponding eigenvectors are

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left(|1,1\rangle + |1,-1\rangle\right), \\ &\frac{1}{\sqrt{2}}\left(|1,1\rangle - |1,-1\rangle\right), \\ &|1,0\rangle . \end{aligned}$$

To see if the Hamiltonian is invariant under time reversal, we assume that A and B are real.

$$\Theta H \Theta^{-1} = A\Theta S_z \Theta^{-1}\Theta S_z \Theta^{-1} + B\left(\Theta S_x \Theta^{-1}\Theta S_x \Theta^{-1} - \Theta S_y \Theta^{-1}\Theta S_y \Theta^{-1}\right).$$

Since (4.4.53 293 Sakurai)

$$\Theta \mathbf{J} \boldsymbol{\Theta}^{-1} = -\mathbf{J}$$

we obtain

$$\Theta H \Theta^{-1} = A(-S_z)^2 + B\left[(-S_x)^2 - (-S_y)^2\right] = H,$$

i.e., the Hamiltonian is invariant under time reversal.

From

$$\Theta\,|j,m\rangle = (-1)^m\,|j,-m\rangle$$

we find

$$\Theta\,(|1,1\rangle + |1,-1\rangle) = -\,(|1,1\rangle + |1,-1\rangle) \quad \text{(odd)},$$
$$\Theta\,(|1,1\rangle - |1,-1\rangle) = +\,(|1,1\rangle - |1,-1\rangle) \quad \text{(even)},$$
$$\Theta\,|1,0\rangle = +\,|1,0\rangle \quad \text{(even)}.$$