7. Sfärisk Symmetri

7.1 Visa att L_x är en Hermitesk operator.

- $L_x = yp_z - zp_y = -i\hbar\left(y\frac{d}{dz} - z\frac{\partial}{\partial y}\right)$
- Hermitesk operator: $\langle u|Av\rangle = \langle A^\dagger u|v\rangle \Leftrightarrow A = A^\dagger$

Vi använder $u(x,y,z)$ och $v(x,y,z)$

$$\langle u|L_x u\rangle = \langle u|yp_z u\rangle - \langle u|zp_y u\rangle = y\langle u|p_z u\rangle - z\langle u|p_y u\rangle =$$

$$= [p_z \,\&\, p_y \text{ Hermitesk}] = y\langle p_z u|u\rangle - z\langle p_y u|u\rangle = [\,y \text{ och } z \in \mathbb{R}\,] =$$

$$= \langle yp_z u|u\rangle - \langle zp_y u|u\rangle = \langle (yp_z - zp_x)u|u\rangle =$$

$$= \langle L_x u|u\rangle \qquad \square$$

7.2 Beräkna följande kommutatorer

a) $[L_x, x] = [yp_z - zp_y, x] = [yp_z, x] - [zp_y, x] =$

$= y[p_z, x] + [y, x]p_z - z[p_y, x] - [z, x]p_y =$

$= 0 + 0 - 0 - 0 = \underline{0}$

$\Rightarrow$ svar: $[L_x, x] = 0$

b) $[L_x, y] = [yp_z - zp_y, y] = [yp_z, y] - [zp_y, y] =$

$= y\underbrace{[p_z, y]}_{0} + \underbrace{[y, y]}_{0}p_z - z\underbrace{[p_y, y]}_{-i\hbar} - \underbrace{[z, y]}_{0}p_y = i\hbar z$

$\Rightarrow$ svar: $[L_x, y] = i\hbar z$

c) $[L_x, z] =$ p.s.s $= -i\hbar y$

d) $[L_z, p_x] = [xp_y - zp_x, p_x] = [xp_y, p_x] - [zp_x, p_x] =$

$= x\underbrace{[p_y, p_x]}_{0} + \underbrace{[x, p_x]}_{i\hbar}p_y - z\underbrace{[p_x, p_x]}_{0} - \underbrace{[z, p_x]}_{0}p_x = i\hbar p_x$

e) $[L_z, p_y] = [xp_x - yp_x, p_y] = [xp_y, p_y] - [yp_x, p_y] =$

$= x\underbrace{[p_y, p_y]}_{0} + \underbrace{[x, p_y]}_{0}p_x - y\underbrace{[p_x, p_y]}_{0} - \underbrace{[y, p_y]}_{i\hbar}p_x =$

$= -i\hbar p_x$

f) $[L_z, p_z] = x\underbrace{[p_y, p_z]}_{0} + \underbrace{[x, p_z]}_{0}p_y - y\underbrace{[p_x, p_z]}_{0} - \underbrace{[y, p_z]}_{0}p_x = 0$

7.3 Beräkna $L_x\phi$ och ΔL_x

$$\phi(\bar{r}) = N\frac{x}{r}e^{-\alpha r}$$

Beräkning av $L_x\phi$

Vi vet att: $(\Delta L_x)^2 = \langle L_x^2\rangle - \langle L_x\rangle^2$

$$L_x = yp_z - zp_y = y\left(-i\hbar\frac{\delta}{\delta z}\right) - z\left(-i\hbar\frac{\delta}{\delta y}\right)$$

$$\frac{\delta}{\delta z}\phi(\bar{r}) = \frac{\delta}{\delta z}\left(N\frac{y}{\sqrt{x^2+y^2+z^2}}\cdot e^{-\alpha(x^2+y^2+z^2)^{1/2}}\right) =$$

$$= Ny\,\frac{-\alpha z - z\cdot\frac{\cdot}{\cdot}}{r^2}e^{-\alpha r} = \frac{N\alpha yz}{r^2}e^{-\alpha r} - \frac{Nyz}{r^3}e^{-\alpha r}$$

$$\frac{\delta}{\delta y}\phi(\bar{r}) = \frac{\delta}{\delta y}\left(N\frac{y}{(x^2+y^2+z^2)^{1/2}}e^{-\alpha(x^2+y^2+z^2)^{1/2}}\right) =$$

$$= N\cdot\frac{(r-\alpha y^2)-\frac{y^2}{r}}{r^2}e^{-\alpha r} =$$

$$= \frac{N}{r}e^{-\alpha r} - \frac{N\alpha y^2}{r^2}e^{-\alpha r} - \frac{Ny^2}{r^3}e^{-\alpha r}$$

$$\Rightarrow L_x\phi = (-i\hbar)\left(-\frac{N\alpha yz}{r^2} - \frac{Nyz}{r^3} - \frac{Nz}{r} + \frac{N\alpha yz}{r^2} - \frac{Nyz}{r^3}\right)e^{-\alpha r} =$$

$$= \frac{i\hbar Nz}{r}e^{-\alpha r}$$

SVAR

Beräkning av $\Delta L_x = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2}$

$$\langle L_x \rangle = \langle \phi \,|\, L_x \phi \rangle = \left\langle N \frac{y}{r} e^{-\alpha r} \,\middle|\, \frac{i\hbar N z}{r} e^{-\alpha r} \right\rangle =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} N\frac{y}{r} \cdot \frac{i\hbar \cdot Nz}{r} e^{-\alpha r} \cdot e^{-\alpha r} r^2 \sin\theta \, dr\, d\theta\, d\phi =$$

$y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} i\hbar N^2 zy e^{-2\alpha r} \cdot \sin\theta \, dr\, d\theta\, d\phi =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} i\hbar N^2 r\cos\theta \cdot r\sin\theta\sin\phi \cdot e^{-2\alpha r} \cdot \sin\theta \, dr\, d\theta\, d\phi =$$

$$= i\hbar N^2 \int_0^{2\pi} \sin\phi \, d\phi \int_0^{\pi} \cos\theta \cdot \sin^2\theta \, d\theta \int_0^{\infty} r^2 dr =$$

$$= i\hbar N^2 \left[-\cos\phi\right]_0^{2\pi} \cdot \underbrace{\left[\frac{\sin^3(\theta)}{3}\right]_0^{\pi}}_{=0} \cdot \left[\frac{1}{r}\right]_0^{\infty} = \underline{0}$$

$$\langle L_x^2 \rangle = \langle \phi \,|\, L_x^2 \phi \rangle = \langle L_x\phi \,|\, L_x\phi \rangle = \hbar^2 N^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{z^2}{r^2} e^{2\alpha r} r^2 \sin\theta \, dr\, d\theta\, d\phi =$$

$$= \hbar^2 N^2 \cdot 2\pi \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \cdot \int_0^{\infty} r^2 e^{-2\alpha r} dr = \ldots = \boxed{\frac{\hbar N^2 \pi}{3\alpha^3}} \quad *$$

Men vad är N^3? Vi normerar ϕ.

Normering

$$1 = \langle \phi | \phi \rangle = N^2 \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{y^2}{r^2} e^{-2\alpha r} r^2 \sin\theta \, dr \, d\theta \, d\phi =$$

$$= \ldots = \frac{N^2 \pi}{3\alpha^3} \iff \boxed{N^2 = \frac{3\alpha^3}{\pi}}$$

Insättning i * ger $\langle L_x^2 \rangle = \frac{\hbar\pi}{3\alpha^3} \cdot \frac{3\alpha^3}{\pi} = \boxed{\hbar}$

$$\Rightarrow \Delta L_x = \sqrt{\hbar^2 - 0} = \boxed{\hbar}$$

svar:

$$\boxed{\begin{aligned} L_x \phi &= \frac{i\hbar N}{r} e^{-\alpha r} \\ \Delta L_x &= \hbar \end{aligned}}$$

7.4 Visa att $\langle L_x \rangle = 0$ i ett egentillstånd till L_z

se sida 152: $i\hbar L_x = [L_y, L_z]$

$$\Leftrightarrow \boxed{L_x = \frac{1}{i\hbar}[L_y, L_z]}$$

Alltså kan vi bestämma $\langle L_x \rangle$ genom att beräkna $[L_y, L_z]$ mha $\boxed{L_z\phi = \hbar m \phi}$

$$\langle L_x \rangle = \langle \phi | L_x \phi \rangle = \left\langle \phi \left| \frac{1}{i\hbar}[L_y, L_z]\phi \right. \right\rangle =$$

$$= \frac{1}{i\hbar}\langle \phi | [L_y, L_z]\phi \rangle =$$

$$= \frac{1}{i\hbar}\langle \phi | (L_yL_z - L_zL_y)\phi \rangle =$$

$$= \frac{1}{i\hbar}\left(\langle \phi | L_xL_z\phi \rangle - \langle \phi | L_zL_y\phi \rangle \right) =$$

($\hbar m\phi$ ovanför $L_xL_z\phi$)

⇕ Eftersom $L_z = L_z^+$

$$= \frac{1}{i\hbar}\left(\hbar m \langle \phi | L_y\phi \rangle - \langle L_z\phi | L_y\phi \rangle \right) =$$

($\hbar m\phi$ under $L_z\phi$)

$$= \frac{1}{i\hbar}\underbrace{\left(\hbar m \langle \phi | L_y\phi \rangle - \hbar m \langle \phi | L_y\phi \rangle \right)}_{=0} = \boxed{0} \quad \square$$

Facit verkar ha $L_z\phi = m\phi$, jag tänkte nog fel...

7.5 Beräkna $\nabla^2\left(r^\ell Y_\ell^m(\theta,\varphi)\right)$

∇^2 finns i formelsamlingen:

$$\nabla^2 = \frac{1}{r}\cdot\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

Och om du läser s. 154 så ser du att:

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{1}{r^2\hbar^2}\hat{L}^2$$

Använd denna.

$$\nabla^2\left(r^\ell Y_\ell^m(\theta,\varphi)\right) = \underbrace{\frac{Y_\ell^m}{r}\frac{\partial^2}{\partial r^2}(r\cdot r^\ell)}_{①} - \underbrace{\frac{1}{r^2\hbar^2}\hat{L}^2(r^\ell Y_\ell^m)}_{②}$$

$$① = \frac{Y_\ell^m}{r}(r^{\ell+1})''_{rr} = \frac{Y_\ell^m}{r}(\ell+1)(r^\ell)'_r = \frac{Y_\ell^m}{r}(\ell+1)\cdot\ell\cdot r^{\ell-1} =$$

$$= Y_\ell^m\cdot\ell(\ell+1)r^{\ell-2}$$

$$② = \frac{\hbar^2\cdot\ell(\ell+1)}{r^2\cdot\hbar^2}r^\ell Y_\ell^m(\theta,\varphi) = Y_\ell^m\cdot\ell(\ell+1)r^{\ell-2}$$

↖ Läs s. 155!

$$① = ② \Rightarrow \nabla^2\left(r^\ell Y_\ell^m(\theta,\varphi)\right) = 0$$

7.6 Visa att f är egenfunktion till $\hat{L}^2$ och $\hat{L}_z$

$$f(\theta,\varphi) = N\sin^n(\theta)\cdot e^{-in\varphi}$$

$$\hat{L}_z f = -i\hbar\frac{\partial f}{\partial\varphi} = -i\hbar N\sin^n(\theta)(-in)e^{-in\varphi} = \underline{-n\hbar f} \quad \square$$

$$\hat{L}^2 f = -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{\tan\theta}\cdot\frac{\partial f}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\right) =$$

$$= -\hbar\Big(n(n-1)N\sin^{n-2}(\theta)\cos^2(\theta) - N n\sin^n(\theta) +$$

$$+ \frac{1}{\tan\theta}Nn\cos(\theta)\sin^{n-1}(\theta) + \frac{N}{\sin^2\theta}\sin^n(\theta)(-in)^2\Big)e^{-in\varphi} =$$

$$= -\hbar N\left(n^2\sin^{n-2}\theta\cos^2\theta - n\sin^{n-2}\cos^2\theta - n\sin^n\theta +\right.$$

$$\left. + n\cos^2\theta\sin^{n-2}\theta - n^2\sin^{n-2}\theta\right)e^{-in\varphi} =$$

$$= -\hbar N\left(n^2\sin^{n-2}\theta(1-\sin^2\theta) - n\sin^n\theta - n^2\sin^{n-2}\theta\right)e^{-in\varphi} =$$

$$= -\hbar N\left(n^2\sin^{n-2}\theta - n^2\sin^n\theta - n\sin^n\theta - n^2\sin^{n-2}\theta\right)e^{-in\varphi} =$$

$$= +\hbar N\left(n(1+n)\sin^n\theta\right)e^{-in\varphi} =$$

$$= \hbar N\cdot n(1+n)\sin^n\theta\, e^{-in\varphi} = \boxed{\hbar n(n+1)\cdot f(\theta,\varphi)} \quad \square$$

7.8 Uttryck funktionen i sfäriska koordinater.

$$\phi(\bar{r}) = N(3x^2 - y^2 + z^2) \cdot f(\bar{r})$$

$$\begin{cases} X = r \cdot \sin\theta \cos\varphi \\ Y = r \sin\theta \sin\varphi \\ Z = r\cos\theta \end{cases}$$

$$x^2 - y^2 + z^2 = \left(3r^2\sin^2\theta\cos^2\varphi - r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\theta\right) =$$

$$= r^2\left(\sin^2\theta\left(3\cos^2\varphi - \underbrace{\sin^2\varphi}_{1-\cos^2\varphi}\right) + \cos^2\theta\right) =$$

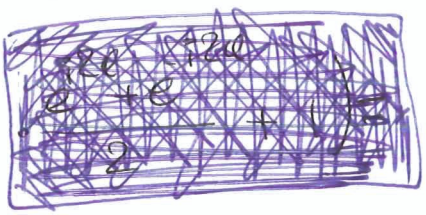
$$= r^2\left(\sin^2\theta\left(3\cos^2\varphi - 1 + \cos^2\varphi\right) + \cos^2\theta\right) =$$

$$= r^2\left(\sin^2\theta\left(4\cos^2\varphi - 1\right) + \cos^2\theta\right) =$$

$$= r^2\left(4\sin^2\theta\cos^2\varphi - \sin^2\theta + \underbrace{\cos^2\theta}_{1-\sin^2\theta}\right) =$$

$$= r^2\left(4\sin^2\theta\cos^2\varphi - \sin^2\theta + 1 - \sin^2\theta\right) =$$

$$= r^2\left(4\sin^2\theta\cos^2\varphi - 2\sin^2\theta + 1\right) = r^2\left(2\sin^2\theta\underbrace{\left(2\cos^2\varphi - 1\right)}_{\cos(2\varphi)} + 1\right) =$$



$$= r^2\left(2\sin^2\theta\cos(2\varphi) + 1\right)$$

$$\Rightarrow \phi(\bar{r}) = Nr^2\left(2\sin^2\theta \cdot \cos(2\varphi) + 1\right) \cdot f(\bar{r})$$

Vilka mätvärden kan erhållas vid mätning av rörelsemoomängdsmomentets z-komponent? slh?

Vi vill uttrycka ϕ som en kombination av en r-beroende funktion och en θ, φ-beroende

$$\phi(\bar{r}) = R(r) \cdot U(\theta, \varphi)$$

Ansats: $U = N(2\sin^2\theta \cdot \cos(2\varphi) + 1)$

L_z agerar ej på den radiella delen, så vi är inte så intresserade av den.

U vill vi skriva som en summa av Y_ℓ^m.

(1) tar hand om ettan i U och (2) ska kunna fixa $\sin^2$ och cos...

(1) $$Y_0^0(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$$

(2) $$Y_2^{\pm 2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} \cdot \sin^2\theta \cdot e^{\pm 2i\varphi}$$

$$Y_0^0 \cdot \sqrt{4\pi} = 1 \quad \text{(fixat)}$$

$$Y_2^2 + Y_2^{-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \left(e^{2i\varphi} + e^{-2i\varphi}\right) =$$

$$= \sqrt{\frac{15}{32\pi}} \sin^2\theta \cdot 2\cos(2\varphi)$$

$$\Rightarrow \left(Y_2^2 + Y_2^{-2}\right) \cdot \sqrt{\frac{32\pi}{15}} = 2\sin^2\theta \cos(2\varphi) \quad \text{(fixat!)}$$

Vi kan nu skriva:

$$U(\bar{r}) = N\left(\sqrt{4\pi}Y_0^0 + \sqrt{\frac{32\pi}{15}}\left(Y_2^2 + Y_2^{-2}\right)\right) =$$

$$= \boxed{N\sqrt{4\pi}\left(Y_0^0 + \sqrt{\frac{8}{15}}\left(Y_2^2 + Y_2^{-2}\right)\right)} \; U(\bar{r})$$

Normering av U

$$1 = \langle U|U\rangle = N^2 \cdot 4\pi\left\langle Y_0^0 + \sqrt{\frac{8}{15}}\left(Y_2^2 + Y_2^{-2}\right) \middle| Y_0^0 + \sqrt{\frac{8}{15}}\left(Y_2^2 + Y_2^{-2}\right)\right\rangle =$$

↶ Bra "trick"

$$= N^2 \cdot 4\pi\left(1 + \frac{8}{15} + \frac{8}{15}\right) = N^2 \cdot 4\pi \cdot \frac{31}{15} \iff \boxed{N = \sqrt{\frac{15}{4\pi 31}}}$$

$$\Rightarrow \boxed{U(\theta,\phi) = \sqrt{\frac{15}{31}}\left(Y_0^0 + \sqrt{\frac{8}{15}}\left(Y_2^2 + Y_2^{-2}\right)\right)}$$

Viktigt! $m = \pm l$. $l = 2, 0$

$$\boxed{L_z\phi = \hbar m\phi, \quad m = -2, 2, 0}$$

SVAR

$m=0$: $P(L_z = 0) = \sqrt{\frac{15}{31}}^2 \cdot 1 = \frac{15}{31}$

$m=2$: $P(L_z = 2\hbar) = \sqrt{\frac{15}{31}}^2 \left(0 + \sqrt{\frac{8}{15}}\right)^2 = \frac{8}{31}$

$m=-2$: $P(L_z = -2\hbar) = \sqrt{\frac{15}{31}}^2 \left(0 + \sqrt{\frac{8}{15}}\right) = \frac{8}{31}$

0 annars!

7.9 En partikel beskrivs vid en viss tidpunkt av:

$$\phi(\bar{r}) = Nf(r)\left(\frac{x}{r} + \frac{z}{r} + 1\right)$$

a) Normera vinkeldelen av funktionen.

$$\begin{cases} x = r\sin\theta\cos\varphi \\ z = r\cos\theta \end{cases}$$

$$\Rightarrow \phi(r) = f(r)\underbrace{N\left(\sin\theta\cos\varphi + \cos\theta + 1\right)}_{U(\theta,\varphi)} = f(r)\cdot U(\theta,\varphi)$$

spara papper

$$1 = \langle U|U\rangle = \int_0^{2\pi}\int_0^{\pi} N^2\left(\sin\theta\cos\varphi + \cos\theta + 1\right)^2 \sin\theta\, d\theta d\varphi = \left[\begin{matrix} s = \sin \\ c = \cos \end{matrix}\right] =$$

$$= N^2 \int_0^{2\pi}\int_0^{\pi} \left(s^2\theta c^2\varphi + s\theta c\varphi c\theta + s\theta c\varphi + s\theta c\varphi c\theta + c^2\theta + c\theta + s\theta c\varphi + c\theta + 1\right) s\theta\, d\theta d\varphi$$

$$= N^2 \cdot \frac{20}{3}\pi \Leftrightarrow N = \sqrt{\frac{3}{20\pi}} \quad \text{(Huvudräkning)}$$

Svar: $U(\theta,\varphi) = \sqrt{\frac{3}{20\pi}}\left(\sin\theta\cos\varphi + \cos\theta + 1\right)$

b) Bestäm $\langle L^2\rangle$

Vi börjar med att skriva om $U(\theta,\varphi)$ mha $Y_\ell^m(\theta,\varphi)$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \Rightarrow \boxed{1 = \sqrt{4\pi}\cdot Y_0^0}$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi} \Rightarrow Y_1^{-1} - Y_1^{1} = \sqrt{\frac{3}{8\pi}}\sin\theta\left(e^{-i\varphi} + e^{i\varphi}\right) =$$

$$= \sqrt{\frac{3}{8\pi}}\sin\theta\cdot 2\cos\theta \Rightarrow \boxed{\sin\theta\cos\theta = \left(Y_1^{-1} - Y_1^{1}\right)\cdot\frac{1}{2}\cdot\sqrt{\frac{8\pi}{3}}}$$

Till sist har vi:

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \iff \boxed{\cos(\theta) = Y_1^0 \cdot \sqrt{\frac{4\pi}{3}}}$$

Vi kan nu skriva om $U(\theta,\varphi)$.

$$U(\theta,\varphi) = \sqrt{\frac{3}{20\pi}}\left(\left(Y_1^{-1} - Y_1^1\right)\cdot\frac{1}{2}\cdot\sqrt{\frac{8\pi}{3}} + Y_1^0\sqrt{\frac{4\pi}{3}} + \sqrt{4\pi}\cdot Y_0^0\right) =$$

$$= \sqrt{\frac{3\cdot 4\pi}{20\pi}}\left(\left(Y_1^{-1} - Y_1^1\right)\cdot\frac{1}{2}\cdot\sqrt{\frac{2}{3}} + Y_1^0\cdot\frac{1}{\sqrt{3}} + Y_0^0\right)$$

Okej, nu ska vi fundera på vad $\langle \hat{L}^2 \rangle$ är...

$\langle \hat{L}^2 \rangle = \langle \phi | \hat{L}^2 \phi \rangle$, okej, vad är $\hat{L}^2\phi$ då?

$$\hat{L}^2\phi = \hbar^2 l(l+1)\phi = \hbar^2 l(l+1)U(\theta,\varphi)\cdot f(r) =$$

$$= \hbar^2 l(l+1)\cdot\sqrt{\frac{3}{5}}\left(\frac{1}{2}\sqrt{\frac{2}{3}}\left(Y_1^{-1} - Y_1^1\right) + \sqrt{\frac{1}{3}}Y_1^0 + Y_0^0\right)$$

Men om $l=0$ blir det ju noll, vi betraktar $l=1$, räknar df ej med Y_0^0...

$$\Rightarrow \hat{L}^2\phi = \hbar^2\cdot 2\sqrt{\frac{3}{5}}\left(\frac{1}{2}\sqrt{\frac{2}{3}}\left(Y_1^{-1} - Y_1^1\right) + \sqrt{\frac{1}{3}}Y_1^0\right) =$$

$$= \hbar\frac{1}{\sqrt{5}}\left(\sqrt{2}\left(Y_1^{-1} - Y_1^1\right) + 2Y_1^0\right)$$

$$\langle \hat{L}^2 \rangle = \langle \phi | \hat{L}^2\phi \rangle = \left(\frac{2}{5} - \frac{2}{5} + \frac{4}{5}\right)\hbar^2 = \boxed{\frac{4\hbar^2}{5}}$$

c) L_z mätv.?

$L_z\phi = \hbar m\phi$

svar: $\pm\hbar$ och 0 eftersom m kan vara $-1, 0, 1$ $= \pm l$.

7.11

$$V(r)=\begin{cases}-V_0 & r<a\\ 0 & r>a\end{cases}, \quad E_1=-\frac{V_0}{2}, \quad \ell=0$$

$u(r)=rR(r)$, Grundtillstånd: $E=-V_0/2$

a) Finn lösningen till radiella Schrödingerekvationen.

$$-\frac{\hbar^2}{2m}\cdot\frac{\partial^2 u}{\partial r^2}+\frac{\hbar^2\ell(\ell+1)}{2mr^2}u+V(r)u=Eu$$

$$\Leftrightarrow \frac{\partial^2 u}{\partial r^2}-\frac{\ell(\ell+1)}{r^2}u-\frac{V(r)\cdot 2m}{\hbar^2}u=-\frac{E\cdot 2m}{\hbar^2}u$$

För $\ell=0$ får vi:

$$\frac{\partial^2 u}{\partial r^2}-\frac{V(r)\cdot 2m}{\hbar^2}u=-\frac{E\cdot 2m}{\hbar^2}u$$

Område I: $r<a \Rightarrow V(r)=-V_0$, $E=-\frac{V_0}{2}$

$$\Rightarrow u''+\frac{2V_0m}{\hbar^2}u=+\frac{V_0}{2}\cdot\frac{2m}{\hbar^2}u$$

$$\Leftrightarrow u''+\frac{V_0m}{\hbar^2}u=0$$

Kända lösningar: $u(r)=A\sin\left(\sqrt{\frac{V_0m}{\hbar^2}}\cdot r\right)+B\cos\left(\sqrt{\frac{V_0m}{\hbar^2}}\cdot r\right)$

RV: $u(0)=0 \Rightarrow B=0$

$$\Rightarrow \boxed{u(r)=A\sin\left(\sqrt{\frac{V_0m}{\hbar^2}}r\right),\ r<a}$$

Område II: $r > a$: $V(r) = 0$, $E = \frac{V_0}{2}$.

$$\Rightarrow u'' - 0 = \frac{V_0}{2} \cdot \frac{2m}{\hbar^2} u$$

$$\Leftrightarrow u'' - \frac{V_0 m}{\hbar^2} u = 0$$

Denna har de kända lösningarna:

$$u(r) = C \cdot e^{-\sqrt{\frac{V_0 m}{\hbar^2}} r} + D \cdot e^{\sqrt{\frac{V_0 m}{\hbar^2}} r}$$

u begränsad $\Rightarrow D = 0$

$$\Rightarrow \boxed{u(r) = C\, e^{-\sqrt{\frac{V_0 m}{\hbar^2}} r} \text{ , } r > a.}$$

Vi har två Passningsvillkor:

1. u kontinuerlig i $r = a$
2. u' kontinuerlig i $r = a$

$$\Rightarrow \begin{cases} A\sin(ka) = Ce^{-ka} \\ kA\cos(ka) = -kCe^{-ka} \end{cases} \text{ , där } k = \sqrt{\frac{V_0 m}{\hbar^2}}$$

$$\Leftrightarrow A\sin(ka) = Ce^{-ka} = -A\cos(ka)$$

$$\Rightarrow \sin(ka) = -\cos(ka) \Leftrightarrow \tan(ka) = -1 \Leftrightarrow \boxed{ka = \frac{3\pi}{4}}$$

$$\Rightarrow \frac{V_0 m}{\hbar^2} a^2 = \left(\frac{3\pi}{4}\right)^2 \Leftrightarrow \boxed{V_0 = \frac{9\pi^2 \hbar^2}{16 m a^2}}$$

SVAR

V_0

b) Bestäm V_{eff} och ett värde på l då det med säkerhet inte finns något bundet tillstånd.

$$V_{eff}(r) = \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)$$

Vi kan inte ha några bundna tillstånd om $V_{eff} \to > 0$ typ.

Vi låter $r \to a \Rightarrow V(r) \to V(a) \simeq -V_0$

$$\Rightarrow \frac{\hbar^2 l(l+1)}{2ma^2} - V_0 > 0 \iff l(l+1) > \frac{2ma^2}{\hbar^2} V_0 = b$$

$$\iff l^2 + l > b \text{ , vi löser } l^2 + l = b \iff l^2 + l - b = 0$$

lösningar: $l = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + b} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2ma^2 V_0}{\hbar^2}}$

Positiva lösningar söks:

$$l > -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2ma^2 V_0}{\hbar^2}}$$