5.1) a) $e_1 \times e_3 = -e_2$, $e_3 \times e_2 = -e_1$

b) $e_2 \times (2e_1 - e_2 + 3e_3) = 2\overbrace{e_2 \times e_1}^{-e_3} - \overbrace{e_2 \times e_2}^{0} + 3\overbrace{e_2 \times e_3}^{e_1} =$

$= 3e_1 - 2e_3$

c)

$$\begin{matrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 & 3 \end{matrix}$$

$u \times v = (0\cdot 3 - 2\cdot 1, 1\cdot 1 - 3\cdot 1, 1\cdot 2 - 0\cdot 1) = (-2,-2,2)$

d)

$$\begin{matrix} 2 & 6 & -3 & 2 & 6 & -3 \\ 0 & 2 & 3 & 0 & 2 & 3 \end{matrix}$$

$u \times v = (6\cdot 3 - 2\cdot(-3), -3\cdot 0 - 2\cdot 3, 2\cdot 2 - 6\cdot 0) = (24, -6, 4)$

5.2) $u \times v = (2,0,4) \times (-1,2,3) = (-8,-10,4) \parallel (-4,-5,2)$

$w = t\cdot(-4,-5,2)$ är vinkelräta mot u och v.

5.3) Normering av u_1 och u_2

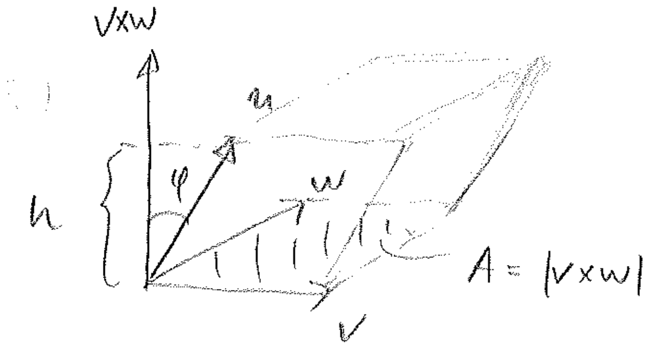
$e_1 = \frac{u_1}{|u_1|} = \frac{1}{\sqrt{6}}(2,-1,1)$ $e_2 = \frac{u_2}{|u_2|} = \frac{1}{\sqrt{2}}(0,1,1)$

$e_3 = e_1 \times e_2 = \frac{1}{\sqrt{6}}(2,-1,1) \times \frac{1}{\sqrt{2}}(0,1,1) = \frac{1}{2\sqrt{3}}(-2,-2,2) = \frac{1}{\sqrt{3}}(-1,-1,1)$

5.4) $(u+v)\cdot((v+w)\times(w+u)) = (u+v)\cdot(v\times w + v\times u + \overset{0}{\overset{\|}{w\times w}} + w\times u) =$

$$= u\cdot(v\times w) + \overbrace{u\cdot(v\times u)}^{=0} + \overbrace{u\cdot(w\times u)}^{=0} + \overbrace{v\cdot(v\times w)}^{=0} + \overbrace{v\cdot(v\times u)}^{=0} + v\cdot(w\times u)$$

$$= u\cdot(v\times w) + \underbrace{v\cdot(w\times u)}_{u\cdot(v\times w)} = 2u\cdot(v\times w)$$



$$u\cdot(v\times w) = |u|\cdot|v\times w|\cdot\cos\varphi = \underbrace{|u|\cdot\cos\varphi}_{\pm h}\cdot\underbrace{|v\times w|}_{A} = A\cdot(\pm h) = \pm V$$

Alltså

$$u\cdot(v\times w) = \begin{cases} V & \text{om } \varphi \le \frac{\pi}{2} \\ -V & \text{om } \frac{\pi}{2} < \varphi \le \pi \end{cases}$$

5.5) $\underbrace{(u+v)}_{\text{I u,v-planet}}\cdot(\underbrace{(u+2v)\times(u-2v)}_{\text{Vinkelrät mot u,v-planet}}) = 0.$

$\cdot = 0$

5.7) $v_1 = (3,-1,2)-(0,-1,1) = (3,0,1)$

$v_2 = (2,3,4)-(0,-1,1) = (2,4,3)$

$$2A = |v_1\times v_2| = |(-4,-7,12)| = \sqrt{4^2+7^2+12^2} = \sqrt{209}$$

$\Rightarrow A = \dfrac{\sqrt{209}}{2}$ a.e.

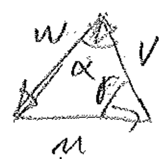
5.6 a) $u+v+w=0$

$u+v=-w$

$$\left.\begin{array}{l}(u+v)\times w = -w\times w = 0 \\ \quad \shortparallel \\ u\times w + v\times w\end{array}\right\} \Rightarrow \begin{array}{c} u\times w = -v\times w \\ \Leftrightarrow \\ w\times u = v\times w \end{array}$$

osv.

b)



$$u\times v = v\times w$$
$$\Leftrightarrow$$
$$|u|\,|v|\cdot\sin\gamma = |v|\cdot|w|\cdot\sin\alpha$$
$$\Leftrightarrow$$
$$\frac{|u|}{\sin\alpha} = \frac{|w|}{\sin\gamma}$$

5.8) a) $v_1 = (2,0,2) - (1,2,1) = (1,-2,1)$

$v_2 = (0,4,-1) - (1,2,1) = (-1,2,-2)$

$$2A = |v_1 \times v_2| = |(0,1,2)| = \sqrt{0^2+1^2+2^2} = \sqrt{5}$$

$$\Rightarrow \quad A = \frac{\sqrt{5}}{2} \text{ a.e.}$$

b) Vi projicerar ner punkterna i xy-planet, dvs sätter z-koordinaten lika med noll

$\hat{v}_1 = (2,0,0) - (1,2,0) = (1,-2,0)$

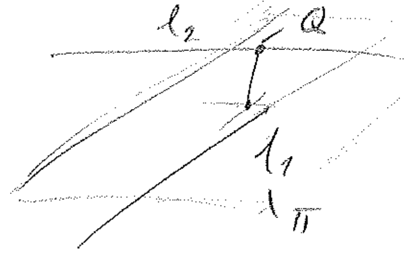
$\hat{v}_2 = (0,4,0) - (1,2,0) = (-1,2,0)$

$$2A = |\hat{v}_1 \times \hat{v}_2| = |(0,0,0)| = 0 \quad \Rightarrow \quad A = 0$$

Anm: Man kan "projicera" v_1 och v_2 också.

5.9) Planet π parallellt med l_2 och innehållande l_1 har riktningsv.

$v_1 = (1,-1,3) \quad v_2 = (1,1,-4)$



och innehåller punkten

$P: (3,1,3)$

$\bar{n} = (1,-1,3) \times (1,1,-4) = (1,7,2)$

$\pi: \quad x + 7y + 2z + d = 0$

$(3,1,3)$ i π ger

$3 + 7 + 6 + d = 0 \Leftrightarrow d = -16$

så

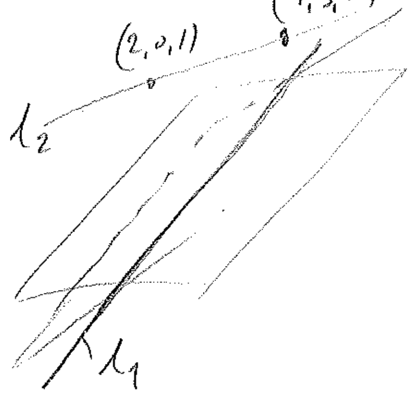
$\pi: \quad x + 7y + 2z - 16 = 0$

Punkt Q på l_2: $\quad Q: (2,-1,-3)$

Avstånd A $l_1 - l_2$ samma som avstånd $Q - \pi$, dvs

$$A = \frac{|2 - 7 - 6 - 16|}{\sqrt{1^2 + 7^2 + 2^2}} = \frac{27}{\underset{27\cdot 2}{\sqrt{54}}} = \frac{\sqrt{27}}{\sqrt{2}} = \frac{3\sqrt{3}}{\sqrt{2}} = \frac{3\sqrt{6}}{2}$$

5.10) "Skärningslinjen"



$$\begin{cases} x-y+z=0 \\ 2x+y-z+3=0 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x-y+z=0 \\ \text{II}-2\text{I} \quad 3y-3z+3=0 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x=-1 \\ y=t-1 \\ z=t \end{cases} \quad - \quad v_1=(0,1,1) \quad P:(-1,-1,0)$$

Riktningsvektor $v_2: (2,0,1)-(-1,3,2) = (3,-3,-1)$

Planet parallellt med l_2 innehållande l_1:

$\bar{n} = (0,1,1) \times (3,-3,-1) = (2,3-3)$

$\pi: \; 2x+3y-3z+d=0$

$(-1,-1,0)$ i π ger

$-2-3+d=0 \Rightarrow d=5$

så $\pi: \; 2x+3y-3z+5=0$

t.ex.

$\Downarrow$

Avstånd A l_1-l_2 samma som avstånd $(2,0,1)-\pi$, dvs

$$A = \frac{|4+0-3+5|}{\sqrt{2^2+3^2+(-3)^2}} = \frac{6}{\sqrt{22}} = \frac{6\sqrt{22}}{22} = \frac{3\sqrt{22}}{11}$$

5.11 a) $u \cdot (v \times w) = w \cdot (u \times v) = (u \times v) \cdot w$ JA

$v \times w = (3, -2, 0) \times (1, 1, 1) = (-2, -3, 5)$

$u \cdot (v \times w) = (-1, 3, 4) \cdot (-2, -3, 5) = 2 - 9 + 20 = 13$

$u \times v = (-1, 3, 4) \times (3, -2, 0) = (8, 12, -7)$

$(u \times v) \cdot w = (8, 12, -7) \cdot (1, 1, 1) = 13$

b) $u \times (v \times w) \neq (u \times v) \times w$ i allmänhet enligt sid 96

$u \times (v \times w) = (-1, 3, 4) \times (-2, -3, 5) = (27, -3, 9)$

$(u \times v) \times w = (8, 12, -7) \times (1, 1, 1) = (19, -15, -4)$.

5.16) $\hat{e}_3 = \frac{1}{\sqrt{3}}(1,-1,-1)$

$\hat{e}_1$ vinkelrät mot $\hat{e}_3$ och $\bar{n}_2 = (1,1,2)$

så $\hat{e}_1$ i riktning

$(1,-1,-1) \times (1,1,2) = (-1,-3,2)$

Tag $\hat{e}_1 = \frac{1}{\sqrt{14}}(1,3,-2)$

$\hat{e}_2 = \hat{e}_3 \times \hat{e}_1 = \frac{1}{\sqrt{42}}(5,1,4)$

Koordinater ...

5.17) $v = (1, a, -1)$, $w = (1,2,3)$, $u \times v = w$.

$u = (x, y, z)$ ger

$$u \times v = (x, y, z) \times (1, a, -1) = (-y - az, z + x, ax - y)$$

$u \times v = w$ ger

$$\begin{cases} -y - az = 1 \\ x + z = 2 \\ ax - y = 3 \end{cases}$$

Gaussa detta "sunkiga" system.

$$\begin{cases} ax - y = 3 \\ x + z = 2 \\ -y - az = 1 \end{cases} \Leftrightarrow \begin{array}{l} \\ a\mathrm{II} - \mathrm{I} \\ \\ \end{array} \begin{cases} ax - y = 3 \\ y + az = 2a - 3 \\ -y - az = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{array}{l} \\ \\ \mathrm{III} + \mathrm{II} \end{array} \begin{cases} ax - y = 3 \\ y + az = 2a - 3 \\ 0 = 2a - 2 \end{cases}$$

Systemet lösbart endast då $a = 1$.

Så om $a \neq 1$ finns inga sådana vektorer u.

Om $a = 1$ så $\begin{cases} x - y = 3 \\ y + z = -1 \end{cases} \Leftrightarrow \begin{cases} x = -t + 2 \\ y = -t - 1 \\ z = t \end{cases}$

dvs $u = (-t + 2, -t - 1, t)$

$\longrightarrow$ Vänd för alternativ lösning.

S.17 alt

$u \times v = w \perp v$ så $w \cdot v = 0$ vilket ger

$$(1, a, -1)\cdot(1,2,3) = 1+2a-3 = 2a-2 = 0$$

$$\Leftrightarrow \quad a = 1$$

Så om $a \neq 1$ finns inga sådana u.

Om $a = 1$ sätt in detta och "lös ut u" som i förra lösningen.

→ S.18

S.19 a) $\overline{P_1P_2} = (1,1,2)$ $\quad \overline{P_1P_3} = (1,2,1)$

$$2A = |(1,1,2)\times(1,2,1)| = |(-3,1,1)| = \sqrt{11}$$

$$\Rightarrow A = \frac{\sqrt{11}}{2}$$

b) $\overline{P_1P_4} = (2,1,1)$

$$\cos\varphi = \frac{(-3,1,1)\cdot(2,1,1)}{|(-3,1,1)|\cdot|(2,1,1)|} = \frac{-4}{\sqrt{11}\cdot\sqrt{6}}$$

$$\Rightarrow$$

$$\varphi = \arccos\left(\frac{-4}{\sqrt{11}\cdot\sqrt{6}}\right)$$

Obs:

Om normalvektorn pekar åt andra hållet (man "kryssar" i omvänd ordning) fås

$$\varphi = \arccos\left(\frac{4}{\sqrt{11}\sqrt{6}}\right)$$

5.18) r.v för l_1: $v_1 = (-2, 1, 2)$

r.v för l_2: $v_2 = (2, 2, 1)$

$v_1 \perp v_2$ ok.

vinkelrät mot!

Normera:

$\hat{e}_1 = \frac{1}{3}(-2, 1, 2)$, $\hat{e}_2 = \frac{1}{3}(2, 2, 1)$

$\hat{e}_3 = \hat{e}_1 \times \hat{e}_2 = \frac{1}{9}(-3, 6, -6) = \frac{1}{3}(-1, 2, -2)$

P:s koordinater

$\hat{x}_1 = (1, 2, 3) \cdot \hat{e}_1 = \frac{1}{3}(-2 + 2 + 6) = 2$, osv...

Anm: Man kan "vända" $\hat{e}_1$ och $\hat{e}_2$ om man vill.

5.20) l_1: $x+y = y+z = \frac{x+z}{2} \iff \begin{cases} x = t \\ y = 0 \\ z = t \end{cases}$ ger $v_1 = (1,0,1)$

l_2: $v_2 = (6,3,2)$

$\bar{n} = v_1 \times v_2 = (1,0,1) \times (6,3,2) = (-3,4,3)$

Planet π innehållande l_1 parallellt med l_2

π: $-3x+4y+3z+d=0$

$(0,0,0)$ i π ger $d=0$ och

π: $-3x+4y+3z=0$

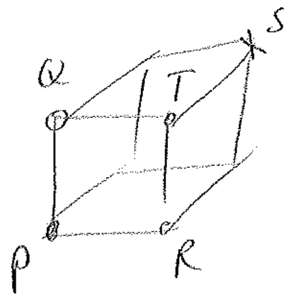
Avstånd $l_1 - l_2$ samma som avstånd $\pi - (1,1,1)$

$$A = \frac{|-3+4+3|}{\sqrt{3^2+4^2+3^2}} = \frac{4}{\sqrt{34}} = \frac{4\sqrt{34}}{34} = \frac{2}{17}\sqrt{34}$$

$$\begin{cases} x+y = y+z \\ x+y = \frac{x+z}{2} \end{cases} \iff \begin{cases} x - z = 0 \\ \frac{1}{2}x + y - \frac{1}{2}z = 0 \end{cases} \iff \begin{cases} x = t \\ y = 0 \\ z = t \end{cases}$$

S.21

$\pi: 2x+y-2z=0$; $P:(1,2,3)$; $Q:(2,4,5)$; $R:(3,0,4)$



$\overline{PQ}=(1,2,2)$ $\quad$ $\overline{QR}=(1,-4,-1)$

$\overline{PR}=(2,-2,1)$

$\overline{PQ}\circ\overline{PR}=0$; $\quad \overline{QR}\circ\overline{PQ}\neq 0$

$\overline{QR}\circ\overline{PR}\neq 0$

$|\overline{PQ}|=|\overline{PR}|=3$

$|\overline{QR}|=\sqrt{18}>3$

$|\overline{PQ}|^2+|\overline{PR}|^2=|\overline{QR}|^2$

$\overline{TS}$ riktad som $\overline{PR}\times\overline{PQ}=(-6,-3,6)\parallel(2,1,-2)$

$\overline{TS}$ av längd 3 så $\overline{TS}=\pm(2,1,-2)$

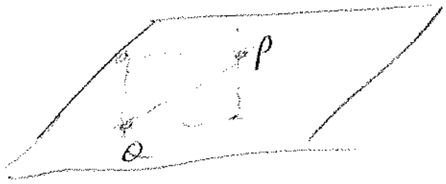
S:s koordinater

$$(1,2,3)+(2,-2,1)+(1,2,2)\pm(2,1,-2)=\begin{cases}(6,3,4)\\(2,1,8)\end{cases}$$

P i π : $2+2-6<0$

så $S=(2,1,8)$

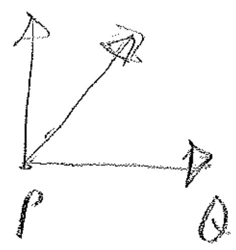
5.24)



Två möjligheter.

$\pi: x+y+z=3 \; ; \; P: (1,1,1) \quad Q: (1,2,0)$

$\overline{PQ} = (0,1,-1)$



Vektor i π vinkelrät mot $\overrightarrow{PQ}$

$\overline{PQ} \times (1,1,1) = (2,-1,-1) \parallel (-2,1,1) \parallel \frac{1}{\sqrt{6}}(-2,1,1)$

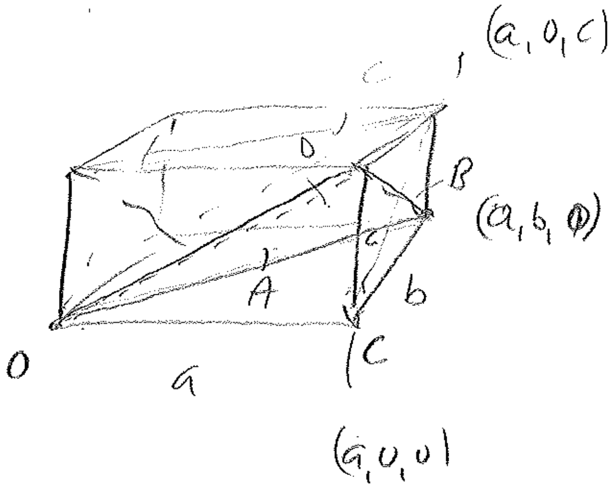
$|\overline{PQ}| = \sqrt{2}$

Möjlig tredje punkt.

$(1,1,1) + \frac{1}{2}(0,1,-1) \pm \frac{\sqrt{3}}{2}\cdot\sqrt{2}\cdot\frac{1}{\sqrt{6}}(-2,1,1)$

$= (1,1,1) + \frac{1}{2}(0,1,-1) \pm \frac{1}{2}(-2,1,1) = \begin{cases}(0,2,1)\\(2,1,0)\end{cases}$

5.25



$D^2 = A^2 + B^2 + C^2$

$D = \left| \frac{1}{2} (a,b,0) \times (a,0,c) \right| = \frac{1}{2} \left| (bc, -ac, -ab) \right|$

$\Rightarrow D^2 = \frac{1}{4} \left((bc)^2 + (ac)^2 + (ab)^2 \right)$

$A^2 = \frac{(ac)^2}{4}$, $B^2 = \frac{(bc)^2}{4}$, $C^2 = \frac{(ab)^2}{4}$

Generalisering av Pyth. sats.