4.1) $|u| = 4$; $|v| = 3$ 

($\frac{\pi}{4}$ rad = 45°)

<u>Def</u> : $u \circ v = |u| \cdot |v| \cdot \cos \varphi$ där φ minsta vinkeln mellan u och v.

a) $u \circ v = 4 \cdot 3 \cdot \cos \frac{\pi}{4} = 4 \cdot 3 \cdot \frac{1}{\sqrt{2}} = 6\sqrt{2}$

b) $(u - 2v) \circ (3u + v) = 3u \circ u + u \circ v - 6 v \circ u - 2 v \circ v =$

$= 3|u|^2 - 5 u \circ v - 2|v|^2 = 3 \cdot 4^2 - 5 \cdot 6\sqrt{2} - 2 \cdot 9 =$

$= 48 - 30\sqrt{2} - 18 = 30(1 - \sqrt{2})$

c) $(u+v) \circ (u+v) = u \circ u + u \circ v + v \circ u + v \circ v =$

$= |u|^2 + 2 u \circ v + |v|^2 = 16 + 12\sqrt{2} + 9 = 25 + 12\sqrt{2}$

d) $|u+v|^2 = (u+v) \circ (u+v) = 25 + 12\sqrt{2}$

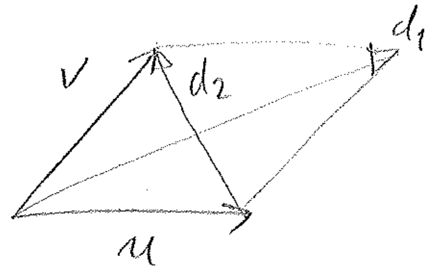
så

$|u+v| = \sqrt{25 + 12\sqrt{2}}$ (ty $|u+v| \geq 0$)

e) $(2u + 3v) \circ (2u - 3v) = 4 u \circ u - 6 u \circ v + 6 v \circ u - 9 v \circ v =$

$= 4|u|^2 - 9|v|^2 = 64 - 81 = -17$

4.4)



$|u| = |v|$ ty romb

a)

$d_1 = u + v$

$d_2 = -u + v$

b) vektorer vinkelräta

$\Leftrightarrow$

deras skalärprodukt är noll

Vi "kollar" d_1 och d_2

$d_1 \circ d_2 = (u+v) \circ (-u+v) = -u \circ u + u \circ v +$

$+ v \circ u + v \circ v = -|u|^2 + |v|^2 = 0$

ty $|u| = |v|$

Alltså $d_1 \perp d_2$.

4.6) $u = x_1 e_1 + x_2 e_2 + x_3 e_3 = (x_1, x_2, x_3)$

Då

$u \circ e_1 = (x_1 e_1 + x_2 e_2 + x_3 e_3) \circ e_1 = x_1 \underbrace{e_1 \circ e_1}_{1} + x_2 \underbrace{e_2 \circ e_1}_{0} + x_3 \underbrace{e_3 \circ e_1}_{0} =$

$= x_1$

dvs $u \circ e_1 = x_1$ ($u \circ e_1$ ger alltså u:s koordinat i e_1:s riktning)

Vi vet $u \circ e_k = k^2$. Alltså

$x_1 = 1^2 = 1$

$x_2 = 2^2 = 4$

$x_3 = 3^2 = 9$

4.9) $e_1' = \frac{1}{3}(1,2,-2)$; $e_2' = \frac{1}{3}(2,1,2)$; $e_3' = \frac{1}{3}(2,-2,-1)$

ON-bas ?

Måste kolla $e_i \cdot e_i = 1$ för $i = 1,2,3$ och

$e_i \cdot e_j = 0$ för $i \neq j$.

$e_1' \cdot e_1' = |e_1|^2 = \frac{1}{9}(1^2 + 2^2 + (-2)^2) = 1$ ok

(analogt för e_2', e_3')

$e_1' \cdot e_2' = \frac{1}{9}(2+2-4) = 0$

$e_1' \cdot e_3' = \frac{1}{9}(2-4+2) = 0$

$e_2' \cdot e_3' = \frac{1}{9}(4-2-2) = 0$ ← måste detta kollas (JA!)

a) Nya koord för u; $u \cdot e_i$

$x_1' = u \cdot e_1' = (1,-1,2) \cdot \frac{1}{3}(1,2,-2) = \frac{1}{3}(1-2-4) = -\frac{5}{3}$

$x_2' = u \cdot e_2' = (1,-1,2) \cdot \frac{1}{3}(2,1,2) = \frac{1}{3}(2-1+4) = \frac{5}{3}$

$x_3' = u \cdot e_3' = (1,-1,2) \cdot \frac{1}{3}(2,-2,-1) = \frac{1}{3}(2+2-2) = \frac{2}{3}$

$u = \left(-\frac{5}{3}, \frac{5}{3}, \frac{2}{3}\right) = \frac{1}{3}(-5,5,2)_{(e')}$

Vänd för b) →

4.9b) $x_1' = u \cdot e_1' = (x_1, x_2, x_3) \cdot \frac{1}{3}(1, 2, -2) = \frac{1}{3}(x_1 + 2x_2 - 2x_3)$

P. s. s.

$x_2' = \quad = \frac{1}{3}(2x_1 + x_2 + 2x_3)$

$x_3' = \quad = \frac{1}{3}(2x_1 - 2x_2 - x_3)$

4.10a) $u = (1, \sqrt{3})$ $v = (0, \sqrt{3})$

$\begin{cases} u \cdot v = 3 \\ |u| = 2 \\ |v| = \sqrt{3} \end{cases}$; $u \cdot v = |u| \cdot |v| \cdot \cos\varphi$

ger

$2 \cdot \sqrt{3} \cdot \cos\varphi = 3$

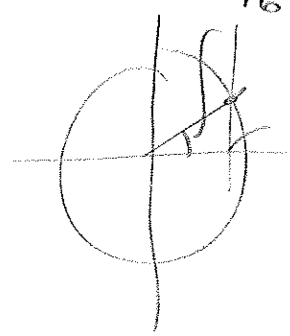
$\Leftrightarrow$

$\cos\varphi = \frac{\sqrt{3}}{2}$

$\Leftrightarrow$

$\varphi = \frac{\pi}{6}$

$\pi/6$ (30°)



4.15) $u = (1, 2, 3)$ $v = (1, 0, 1)$ (finns två vinkelräta av längd 1).

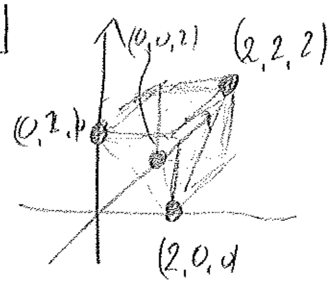
Låt $w = (x, y, z)$, Vi kräver att

$\begin{cases} w \cdot u = 0 \\ w \cdot v = 0 \\ |w|^2 = 1 \end{cases} \Leftrightarrow \begin{cases} x + 2y + 3z = 0 \\ x + z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases} \Leftrightarrow \begin{cases} y = -z \\ x = -z \\ (-z)^2 + (-z)^2 + z^2 = 1 \end{cases}$

(olinjärt)

så $3z^2 = 1 \Rightarrow z = \pm \frac{1}{\sqrt{3}}$, dvs $w = \frac{1}{\sqrt{3}}(-1, -1, 1)$ eller $w = \frac{1}{\sqrt{3}}(1, 1, -1)$

4.14



Kolatom i (1,1,1).

$$V_1 = (2,2,2) - (1,1,1) = (1,1,1)$$

$$V_2 = (2,0,0) - (1,1,1) = (1,-1,-1)$$

$$\cos\alpha = \frac{V_1 \cdot V_2}{|V_1|\cdot|V_2|} = \frac{-1}{\sqrt{3}\cdot\sqrt{3}} = -\frac{1}{3}$$

$$\alpha \approx 109{,}5^\circ$$

4.16) Man verifierar att $v_1 \circ v_2 = 0$ och $|v_1| = |v_2| = 1$.

$$v_3 = v_1 \times v_2 = \frac{1}{3}(-2,-1,2) \times \frac{1}{3}(1,2,2) = \frac{1}{9}(-6,6,-3) = \frac{1}{3}(-2,2,-1)$$

$u = x_1 v_1 + x_2 v_2 + x_3 v_3$ där $x_i = u \circ v_i$

så

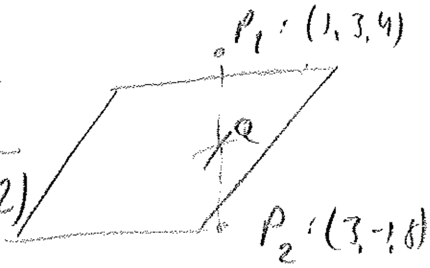
$$x_1 = (1,1,1) \circ \frac{1}{3}(-2,-1,2) = -\frac{1}{3}$$

$$x_2 = (1,1,1) \cdot \frac{1}{3}(1,2,2) = \frac{5}{3}$$

$$x_3 = (1,1,1) \circ \frac{1}{3}(-2,2,-1) = -\frac{1}{3}$$

4.42) Planets normal:

$$\overline{P_1P_2} = (3,-1,8) - (1,3,4) = (2,-4,4) \parallel \underset{\sim}{\overset{\bar{n}}{}}(1,-2,2)$$



Mittpunkt Q:

$$Q = \frac{(3,-1,8) + (1,3,4)}{2} = \frac{(4,2,12)}{2} = (2,1,6)$$

$$\pi : \quad x - 2y + 2z + d = 0$$

$(2,1,6)$ i π ger

$$2 - 2 + 12 + d = 0 \iff d = -12$$

så $\pi : \quad x - 2y + 2z - 12 = 0$

4.17) $\Pi_1: x+y+3z+6=0$
$\Pi_2: 2x+y-2z-10=0$ } skärningslinje l_1

l_2 genom $(6,0,1)$ och $(-4,4,-7)$.

Linjerna på parameterform

$$l_1: \begin{cases} x+y+3z+6=0 \\ 2x+y-2z-10=0 \end{cases} \Leftrightarrow \begin{cases} x+y+3z+6=0 \\ -y-8z-22=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=5t+16 \\ y=-8t-22 \\ z=t \end{cases} \quad \left(\begin{aligned} x=-y-3z-6 &= 8t+22-3t-6 \\ &= 5t+16 \end{aligned} \right)$$

$$l_2: \bar{v} = (-4,4,-7)-(6,0,1) = (-10,4,-8) \parallel (-5,2,-4)$$

$$\begin{cases} x = 6-5s \\ y = 0+2s \\ z = 1-4s \end{cases} \quad (t \text{ upptaget})$$

(1, 2, -3)

"$l_1 = l_2$"

$$\begin{cases} 5t+16 = 6-5s \\ -8t-22 = 2s \\ t = 1-4s \end{cases} \Leftrightarrow \begin{cases} 5t+5s=-10 \\ -8t-2s=22 \\ t+4s=1 \end{cases} \Leftrightarrow \overset{I/5}{\begin{cases} t+s=-2 \\ -8t-2s=22 \\ t+4s=1 \end{cases}}$$

$$\Leftrightarrow \begin{cases} t+s=-2 \\ 6s=6 \\ 3s=3 \end{cases} \Leftrightarrow \begin{cases} t=-3 \\ s=1 \end{cases}$$ lösning finns så linjerna skär.

$s=1$ ger $\begin{cases} x=1 \\ y=2 \\ z=-3 \end{cases}$ som skärningspunkt $\longrightarrow$ VÄND PÅ VÄNDBL

4.17) Riktningsvektorer för linjerna

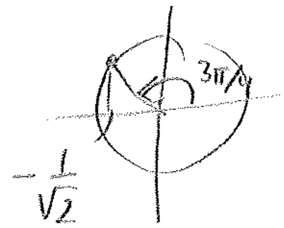
$$v_1 = (5,-8,1) \quad v_2 = (-5,2,-4)$$

$$v_1 \cdot v_2 = -25-16-4 = -45$$

$$|v_1| = \sqrt{25+64+1} = \sqrt{90}$$

$$|v_2| = \sqrt{25+4+16} = \sqrt{45}$$

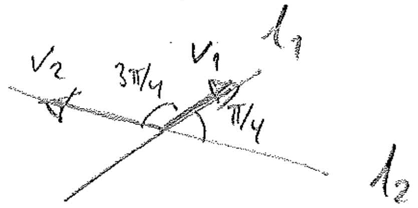
$$\cos\varphi = \frac{v_1 \cdot v_2}{|v_1|\cdot|v_2|} = \frac{-45}{\sqrt{90}\cdot\sqrt{45}} = \frac{-1}{\sqrt{2}}$$



så $\varphi = \frac{3\pi}{4}$

Minsta vinkel mellan linjerna är då

$$\pi - \frac{3\pi}{4} = \frac{\pi}{4}$$



4.18) a) Linje genom $(1,2)$ med $\bar{n}=(-3,4)$

$$-3x+4y+c=0$$

$(1,2)$ på linjen ger

$$-3\cdot 1+4\cdot 2+c=0 \iff c=-5$$

så

$$-3x+4y-5=0$$

Varför fungerar det så?

Om (x,y) är godt punkt på linjen så är vektorn $\bar{n}=(-3,4)$ vinkelrät mot $\bar{v}=(x-1,\ y-2)$ dvs

$$0=\bar{n}\cdot\bar{v}=-3(x-1)+4(y-2)$$

$$\iff$$

$$-3x+4y-5=0$$

b) Planet genom $(1,0,1)$ med $\bar{n}=(-2,3,1)$

$\pi:\ -2x+3y+z+d=0$

$(1,0,1)$ i π ger

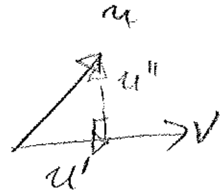
$$-2\cdot 1+3\cdot 0+1+d=0 \iff d=1$$

Alltså $\pi:\ -2x+3y+z+1=0$

4.22 a) u:s proj u' på $v = (1,4,0)$

Proj formel

$$u'' = \frac{u \cdot v}{|v|^2} \cdot v$$



$$u' = \frac{(1,-1,1)\cdot(1,4,0)}{|(1,4,0)|^2} \cdot (1,4,0) =$$

$$= \frac{-3}{17} \cdot (1,4,0)$$

$$u'' = u - u' = (1,-1,1) + \frac{3}{17}(1,4,0) =$$

$$= \frac{1}{17}(20,-5,17)$$

b) $l: \ \frac{x-1}{2} = 2y = z+1$

på parameterform (sätt $y=t$)

$$\begin{cases} x = 4t+1 \\ y = t \\ z = 2t-1 \end{cases}$$

har riktningsvektor $v = (4,1,2)$

$$u' = \frac{(1,-1,1)\cdot(4,1,2)}{|(4,1,2)|^2} \cdot (4,1,2) = \frac{5}{21}(4,1,2)$$

$$u'' = u - u' = (1,-1,1) - \frac{5}{21}(4,1,2) = \frac{1}{21}(1,-26,11)$$

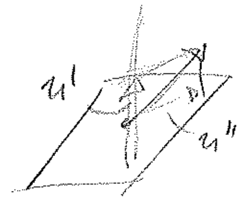
$\longrightarrow$

4.22 c) $\pi: x + 2y - z = 1$

har normal $\bar{n} = (1,2,-1)$

$$u' = \frac{(1,-1,1)\cdot(1,2,-1)}{|(1,2,-1)|^2}\cdot(1,2,-1) = \frac{-2}{6}(1,2,-1) = -\frac{1}{3}(1,2,-1)$$

$$u'' = (1,-1,1) + \frac{1}{3}(1,2,-1) = \frac{1}{3}(4,-1,2)$$



4.24) $\pi: 2x - y + 3z = 0$; $\bar{n} = (2,-1,3)$

$$u' = \frac{(1,-1,2)\cdot(2,-1,3)}{|(2,-1,3)|^2}\cdot(2,-1,3) = \frac{9}{14}\cdot(2,-1,3)$$

$$u'' = (1,-1,2) - \frac{9}{14}(2,-1,3) = \frac{1}{14}(-4,-5,1)$$

Avståndet $|u'| = \frac{9}{14}|(2,-1,3)| = \frac{9}{14}\cdot\sqrt{14} = \frac{9}{\sqrt{14}}$

4.25) Avståndsformel punkt-plan $A = \frac{|ax_0+by_0+cz_0+d|}{(a^2+b^2+c^2)^{1/2}}$

a) $\frac{|1+0-1-2|}{\sqrt{1^2+1^2+(-1)^2}} = \frac{2}{\sqrt{3}}$

b) $\frac{|2+2+1-3|}{\sqrt{2^2+2^2+(-1)^2}} = \frac{2}{3}$

c) Riktningsvektorer för π:

$v_1 = (0,0,1) - (1,0,0) = (-1,0,1)$

$v_2 = (1,1,0) - (1,0,0) = (0,1,0)$

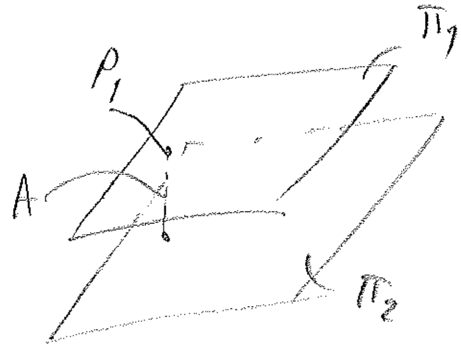
$\bar{n} = v_1 \times v_2 = (-1,0,-1) \parallel (1,0,1)$

$\pi: x + z - 1 = 0$

$$\frac{|1+1-1|}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}$$

4.26) $\pi_1: x-2y-4z-5=0$, $\bar{n}=(1,-2,-4)$
$\pi_2: x-2y-4z-6=0$, $\bar{n}=(1,-2,-4)$ } Planen parallella (annars avstånd automatiskt 0)

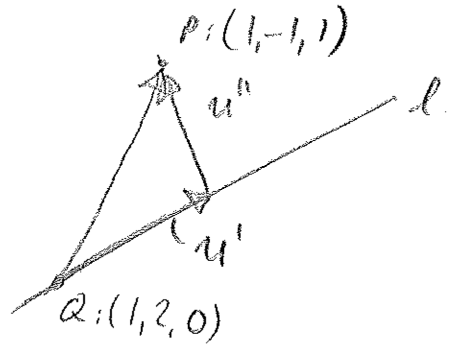
En punkt i π_1



$P_1: (5,0,0)$

Avstånd $\pi_1 - \pi_2$ är samma som avstånd $P_1 - \pi_2$, så

$$A = \frac{|5-6|}{\sqrt{1^2+(-2)^2+(-4)^2}} = \frac{1}{\sqrt{21}}$$



4.29 a) Punkt Q på

$$l: \begin{cases} x = 1+t \\ y = 2-t \\ z = 2t \end{cases} \quad \left(v=(1,-1,2)\right)$$

$Q=(1,2,0)$

$\overline{QP} = (1,-1,1)-(1,2,0) = (0,-3,1)$

$u=\overline{QP}$ proj på l

$$u' = \frac{(0,-3,1)\cdot(1,-1,2)}{|(1,-1,2)|^2}\cdot(1,-1,2) = \frac{5}{6}(1,-1,2)$$

$$u'' = u-u' = (0,-3,1)-\frac{5}{6}(1,-1,2) = \frac{1}{6}(-5,13,-4)$$

$$A = |u''| \; \frac{1}{6}\sqrt{25+169+16} = \frac{1}{6}\sqrt{\underset{2\cdot3\cdot5\cdot7}{210}} = \sqrt{\frac{35}{6}}$$

b) $\longrightarrow$

4.29 b) $P:(1,0,1)$

Punkt på linjen $Q:(1,0,0)$

Riktningsvektor $v=(2,2,1)$

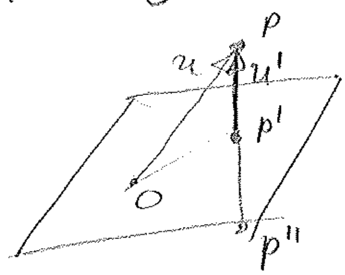
$$u=\overline{QP}=(1,0,1)-(1,0,0)=(0,0,-1)$$

$$u'=\frac{(0,0,-1)\cdot(2,2,1)}{|(2,2,1)|^2}\cdot(2,2,1)=\frac{-1}{9}(2,2,1)$$

$$u''=(0,0,-1)+\frac{1}{9}(2,2,1)=\frac{1}{9}(2,2,-8)=\frac{2}{9}(1,1,-4)$$

$$A=|u''|=\frac{2}{9}(1^2+1^2+4^2)^{1/2}=\frac{2}{9}\cdot\sqrt{18}=\frac{2\sqrt{2}}{\sqrt{9}}=\frac{\sqrt{8}}{3}$$

4.31) $\pi: 2x-y+2z=0$, $\bar{n}=(2,-1,2)$



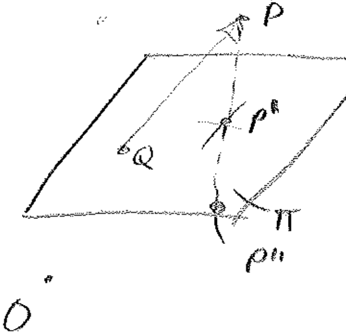
$u=(1,0,1)$ (eftersom O = origo)

$$u'=\frac{(1,0,1)\cdot(2,-1,2)}{|(2,-1,2)|^2}\cdot(2,-1,2)=\frac{4}{9}(2,-1,2)$$

$$OP'=u-u'=(1,0,1)-\frac{4}{9}(2,-1,2)=\frac{1}{9}(1,4,1)=P'$$

$$OP''=u-2u'=\qquad=\frac{1}{9}(-7,8,-7)=P''$$

4.33) Punkt Q i π



$Q: (0,0,1)$

$$u = \overline{QP} = (1,1,6) - (0,0,1) = (1,1,5)$$

$$u' = \frac{(1,1,5)\cdot \overbrace{(2,2,1)}^{\vec{n}}}{|(2,2,1)|^2} \cdot (2,2,1) = \frac{9}{9}(2,2,1) = (2,2,1)$$

$$OP'' = OP + PP'' = (1,1,6) - 2(2,2,1) = (-3,-3,4)$$

4.35) $|u|=5$, $|v|=2$, $|w|=1$, vinkel mellan u och w $\frac{\pi}{3}$

Vill $|u+2v+w| = |u+v+2w|$

$$|u+2v+w|^2 = (u+2v+w)\cdot(u+2v+w) = |u|^2+4|v|^2+|w|^2$$
$$+\ 4u\cdot v + 2u\cdot w + 4v\cdot w$$

$$|u+v+2w|^2 = (u+v+2w)\cdot(u+v+2w) = |u|^2+|v|^2+4|w|^2$$
$$+\ 2u\cdot v + 4u\cdot w + 4v\cdot w$$

Alltså

$$5^2+4\cdot 2^2+1^2+4u\cdot v+2u\cdot w = 5^2+2^2+4\cdot 1^2+2u\cdot v+4u\cdot w$$

$$\Leftrightarrow$$

$\left(u\cdot w = \frac{5}{2}\right)$

$$16+1+4u\cdot v+5 = 4+4+2u\cdot v+10$$

$$22+2u\cdot v = 18$$

$$2u\cdot v = -4$$

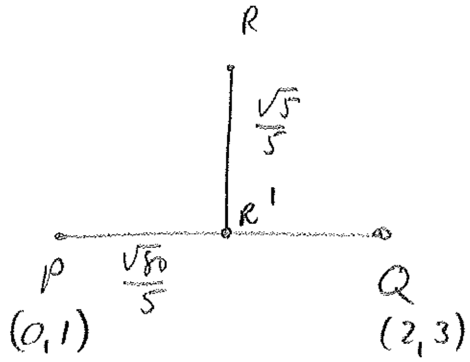
$$u\cdot v = -2$$

$$5\cdot 2\cdot \cos\alpha = -2 \qquad -\frac{1}{5}$$

$$\cos\alpha = -\frac{1}{5}$$

$$\alpha = \arccos\left(-\frac{1}{5}\right)$$

4.36)



$\overrightarrow{PQ} = (2,2)$ $\quad \overrightarrow{RR'}$ parallell med $\frac{1}{\sqrt{2}}(1,-1)$

$\frac{1}{\sqrt{2}}(1,1)$

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR'} + \overrightarrow{R'R} = (0,1) + \frac{\sqrt{80}}{5} \cdot \frac{1}{\sqrt{2}}(1,1) \pm \frac{\sqrt{5}}{5} \cdot \frac{1}{\sqrt{2}}(1,-1) =$$

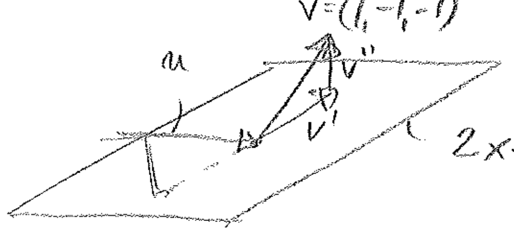
$$= (0,1) + \frac{2\sqrt{2}}{\sqrt{5}}(1,1) \pm \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{2}}(1,-1) =$$

$$= (0,1) + \frac{4}{\sqrt{10}}(1,1) \pm \frac{1}{\sqrt{10}}(1,-1) =$$

$$= \text{i)} \left(\frac{5}{\sqrt{10}}, 1 + \frac{3}{\sqrt{10}}\right)$$

$$\text{ii)} \left(\frac{3}{\sqrt{10}}, 1 + \frac{5}{\sqrt{10}}\right)$$

4.37)



$2x+y-3z=0$; $n=(2,1,-3)$

$\bar{v}$:s proj på $\bar{n}$.

$$v'' = \frac{v \cdot n}{|n|^2} \cdot n = \frac{2-1+3}{14} \cdot (2,1,-3) = \frac{2}{7}(2,1,-3)$$

$(-1,-11,5)$

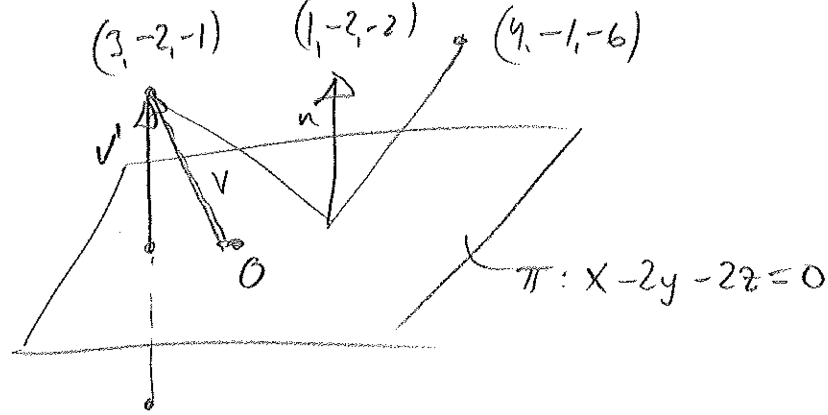
$$v' = v - v'' = (1,-1,-1) - \frac{4}{14}(2,1,-3)$$

$$u = -v'' + v' = -v'' + v - v'' = (1,-1,-1) - \frac{4}{7}(2,1,-3)$$

$$= \frac{1}{7}(-1,-11,5)$$

alt: $u = v - 2v'' = (1,-1,-1) - \frac{4}{7}(2,1,-3) = \frac{1}{7}(-1,-11,5)$

4.40)



Vi speglar $(3,-2,-1)$ i planet:

v:s proj på n: $\frac{v \cdot n}{|n|^2} \cdot n = \frac{(3,-2,-1)\cdot(1,-2,-2)}{3^2} \cdot (1,-2,-2) =$

$= \frac{9}{9}(1,-2,-2) = (1,-2,-2)$

spegelbild: $v - 2v' = (3,-2,-1) - 2(1,-2,-2) = (1,2,3)$

Linje genom $(1,2,3)$ och $(4,-1,-6)$:

$u = (4,-1,-6) - (1,2,3) = (3,-3,-9) \parallel (1,-1,-3)$

$l: (1,2,3) + t(1,-1,-3)$

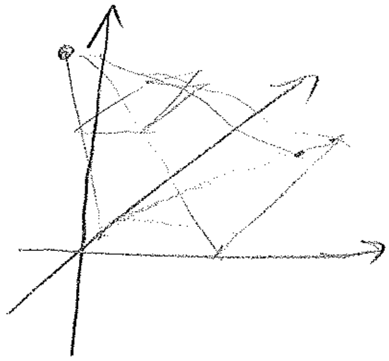
$(2,1,0)$

Skärning linje - plan

$1 + t - 2(2-t) - 2(3-3t) = 0 \Leftrightarrow -9 + 9t = 0 \Leftrightarrow t = 1$

$t = 1$ ger $\underline{\underline{(2,1,0)}}$

4.43)



Sol: $v = (-2, 4, 1)$

Horn: $(1,1,1)$ P_1

$(3,-2,7)$ P_2

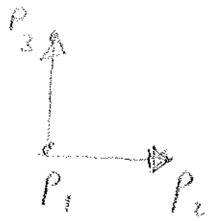
$(4,7,3)$ P_3

Fjärde hörnet

$P_1P_2 = (2,-3,6)$

$P_1P_3 = (3,6,2)$

$\overline{P_1P_2} \circ \overline{P_1P_3} = 0$ så vinkelräta



$P_4 = P_2 + \overline{P_1P_3} = (3,-2,7) + (3,6,2) = (6,4,9)$

Linjen l_1 genom P_1 i solens riktning

$$l_1: \begin{cases} x = 1-2t \\ y = 1+4t \\ z = 1+t \end{cases}$$

$z=0$ ger $t=-1$ och $(3,-3,0)$

$$l_4: \begin{cases} x = 6-2t \\ y = 4+4t \\ z = 9+t \end{cases}$$

$z=0$ ger $t=-9$ och $(24,-32,0)$

osv...