

KAPITEL 11

11.1

Beräkna de symboliska integralerna

$$a) \int_{-\infty}^{\infty} \cos t \delta(t) dt = \cos(0) = 1$$

$$b) \int_{-\infty}^{\infty} e^{-t^2} \delta(t-2) dt = e^{-2^2} = e^{-4}$$

11.3

$$\begin{aligned} a) (e^{-t} \Theta(t))' &= (e^{-t})' \Theta(t) + e^{-t} \Theta'(t) = \\ &= -e^{-t} \Theta(t) + e^{-t} \Theta'(t) = \\ &= -e^{-t} \Theta(t) + \delta(t) \end{aligned}$$

$$b) (\Theta(t) - \Theta(t-1))' = \delta(t) - \delta(t-1)$$

11.5

Förenkla produkterna

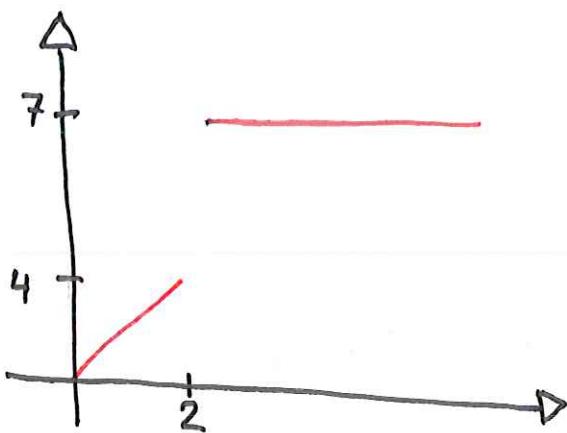
a) $t \delta(t) = \boxed{0}$

b) $t \delta(t-1) = \boxed{\delta(t-1)}$

c) $e^{-t} \delta(t-2) = \boxed{e^{-2} \delta(t-2)}$

d) $\sin t \delta(t-\pi) = \boxed{0}$

11.6



a) $\boxed{2t \Theta(t) + (7-2t) \Theta(t-2)}$

b) $\left(2t \Theta(t) + (7-2t) \Theta(t-2)\right)' =$

$$= 2 \Theta(t) + 2t \delta(t) + -2 \Theta(t-2) + (7-2t) \delta(t-2) =$$

$$= 2 \Theta(t) + 0 - 2 \Theta(t-2) + 3 \delta(t-2) =$$

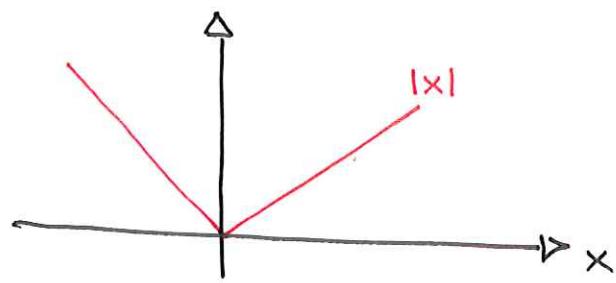
$$= \boxed{3\delta(t-2) + 2(\Theta(t) - \Theta(t-2))}$$

c) $\boxed{\text{Samma svar som i b), tänk lite bärn.}}$

11.7

Beräkna distributionsderivaterna f' och f''

$$f(x) = |x|$$



Jag skriver $|x|$ som en stegfunktion:

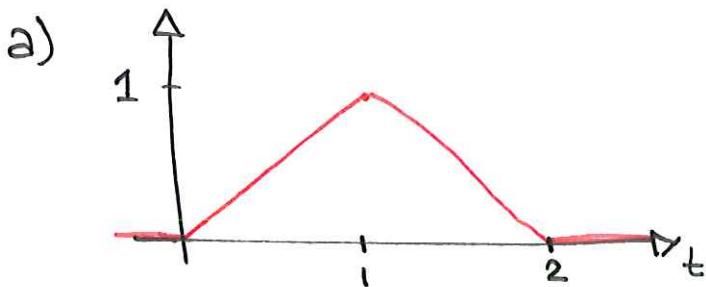
$$|x| = x(-\Theta(-x) + \Theta(x)) \quad (= \begin{cases} -x & \text{då } x < 0 \\ x & \text{då } x > 0 \end{cases})$$

$$f'(x) = (-\Theta(-x) + \Theta(x)) + x(-\Theta'(-x) + \Theta'(x)) =$$

$$= \Theta(x) - \Theta(-x) + x\delta(x) - x\delta(-x) =$$

11.10

Skriv mha stegfunktioner.



SUAR

$$f(t) = t(\Theta(t) - \Theta(t-1)) + (2-t)(\Theta(t-1) - \Theta(t-2))$$

b) Bestäm $f''(t)$

$$\begin{aligned} f'(t) &= \Theta(t) - \Theta(t-1) + t(f(t) - f(t-1)) + \\ &\quad - (\Theta(t-1) - \Theta(t-2)) + (2-t)(\delta(t-1) - \delta(t-2)) = \end{aligned}$$

$$= \Theta(t) - \Theta(t-1) - \cancel{\delta(t-1)} - \Theta(t-1) + \Theta(t-2) +$$

$$+ \cancel{\delta(t-1)} =$$

$$= \underline{\Theta(t)} - 2\Theta(t-1) + \Theta(t-2)$$

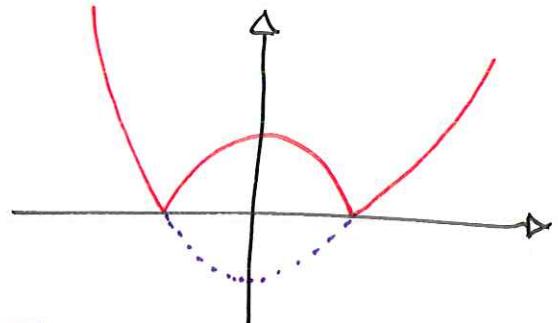
SUAR

$f''(t) = \delta(t) - 2\delta(t-1) + \delta(t-2)$

11.11

Beräkna $f''(t)$

$$f(t) = |1-t^2| = \begin{cases} 1-t^2 & \text{om } 1-t^2 > 0 \Leftrightarrow t^2 < 1 \\ -(1-t^2) & \text{om } 1-t^2 < 0 \Leftrightarrow t^2 > 1 \end{cases}$$



$$f(t) = (1-t^2)(\Theta(t+1) - \Theta(t-1)) + \\ -(1-t^2)(1-\Theta(t+1) + \Theta(t-1)) =$$

$$= \boxed{(1-t^2)(2\Theta(t+1) - 2\Theta(t-1) - 1)}$$

$$f'(t) = -2t(2\Theta(t+1) - 2\Theta(t-1) - 1) + \\ + (1-t^2)(2\delta(t+1) - 2\delta(t-1)) =$$

$$\Leftrightarrow = \boxed{-2t(2\Theta(t+1) - 2\Theta(t-1) - 1)}$$

$$f''(t) = -2(2\Theta(t+1) - 2\Theta(t-1) - 1) +$$

$$-2t(2\delta(t+1) - 2\delta(t-1)) =$$

$$= \boxed{-2(2\Theta(t+1) - 2\Theta(t-1) - 1) + \\ + 2 \cdot 2\delta(t+1) + 2 \cdot 2\delta(t-1)}$$

$$f'''(t) = -4\delta(t+1) + 4\delta(t-1) + 4\delta'(t+1) + 4\delta'(t-1) =$$

$$= \boxed{4(\delta'(t+1) - \delta(t+1) + \delta'(t-1) + \delta(t-1))}$$

11.12

Förenkla produkterna

a) $t \cdot \delta'(t)$

Enl. sats 11.6: $f(t)\delta'(t) = f(0)\delta'(t) - f'(0)\delta(t)$

$\Rightarrow t \cdot \delta'(t) = 0 \cdot \delta'(t) - 1 \cdot \delta(t) = \boxed{-\delta(t)}$ SVAR

b) $e^{-t}\delta'(t) = e^0 \cdot \delta'(t) - (-1)e^0 \delta(t) = \boxed{\delta'(t) + \delta(t)}$ SVAR

c) $\sin t \delta''(t) =$

$f(t) \cdot \delta''(t) = (f(t)\delta'(t))' - f'(t)\delta'(t)$

$\Rightarrow (\sin t \cdot \delta''(t)) = (\sin t \delta'(t))' - \cos t \delta'(t) =$

$= (0 \cdot \delta'(t) - 1 \cdot \delta(t))' - (1 \cdot \delta'(t) + 0 \cdot \delta(t)) =$

$= -\delta'(t) - \delta'(t) = \boxed{-2\delta'(t)}$ SVAR

d) $t \cdot \delta'(t+1) = -1 \cdot \delta'(t+1) - \delta(t+1) = \boxed{-\delta'(t+1) - \delta(t+1)}$

11.13

Bestäm alla primitiva funktioner (=distributioner)

a) $\int (\delta(t) - t) dt = \Theta(t) - \frac{1}{2}t^2 + C$

b) $\int (e^{2t} \Theta(t)) dt = \left(\frac{1}{2} e^{2t} - \frac{1}{2} e^0 \right) \Theta(t) = \frac{1}{2} (e^{2t} - 1) \Theta(t)$ SVAR

c) $\int (\sin t \Theta(t) + \delta(t-2)) dt = (-\cos t - \cos(0)) \Theta(t) + \Theta(t-2) =$
 $= (1 - \cos t) \Theta(t) + \Theta(t-2)$ SVAR

d) $\int \delta''(t+1) dt = \delta'(t+1) + C$ SVAR

e) $\int t \Theta(t) dt = \left(\frac{1}{2} t^2 + 0 \right) \Theta(t) = \frac{1}{2} t^2 \Theta(t) + C$ SVAR

f) $\int t \cdot \Theta(t-3) dt = \left(\frac{1}{2} t^2 - \frac{1}{2} 3^2 \right) \Theta(t-3) = \frac{1}{2} (t^2 - 9) \Theta(t-3) + C$ SVAR

g) $\int (\delta_{-2}(t) - \delta_1(t)) dt = \Theta(t+2) - \Theta(t-1) + C$ SVAR

h) $\int (x-1) \Theta(x-1) dx = \left(\frac{1}{2} (x-1)^2 - \frac{1}{2} (1-1)^2 \right) \Theta(x-1) = \frac{1}{2} (x-1)^2 \Theta(x-1) + C$ SVAR

11.14

Bestäm alla distributionslösningar

$$y''(x) = \delta(x) + x$$

$$y'(x) = \Theta(x) + \frac{1}{2}x^2 + C$$

SVAR

$$y(x) = x \cdot \Theta(x) + \frac{1}{6}x^3 + Cx + D$$

11.15

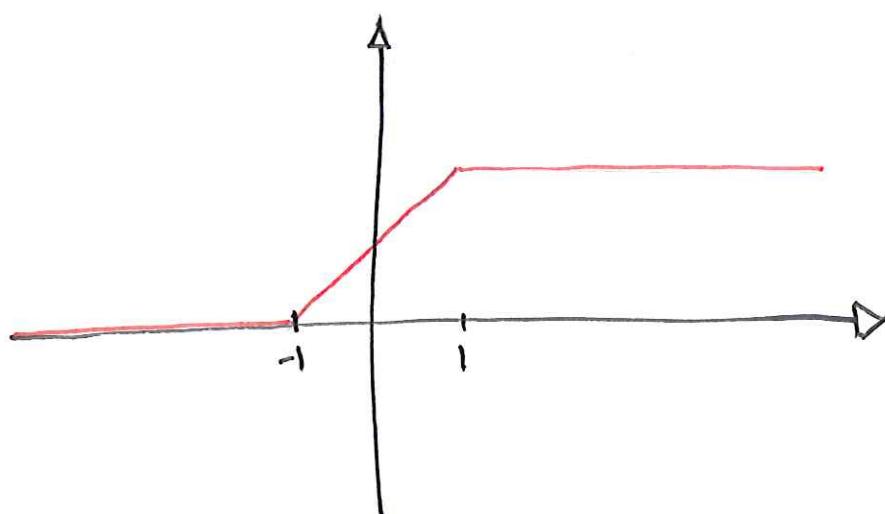
Bestäm den distribution y som uppfyller

$$\begin{cases} y''(t) = \delta(t+1) - \delta(t-1) \\ y(t) = 0 \text{ då } t < -1 \end{cases}$$

$$y'(t) = \Theta(t+1) - \Theta(t-1) + C$$

$$y(t) = (t+1)\Theta(t+1) - (t-1)\Theta(t-1) + Ct + D$$

Eftersom $y(t) = 0$ då $t < -1$
och de två första termerna
 $= 0$ då $t < -1$ så betyder
det att $C = D = 0$



11.16

Bestäm $U(x)$

$$\left\{ \begin{array}{l} \frac{d^4 U}{dx^4} = \delta'(x-1) \\ U(0) = U(2) = 0 \\ U''(0) = U''(2) = 0 \end{array} \right.$$

$$\frac{d^3 U}{dx^3} = \delta(x-1) + A$$

$$\frac{d^2 U}{dx^2} = \Theta(x-1) + Ax + B$$

$$\frac{d U}{dx} = (x-1)\Theta(x-1) + A \frac{1}{2}x^2 + Bx + C$$

$$U(x) = \frac{1}{2}(x-1)^2\Theta(x-1) + A \frac{1}{6}x^3 + B \frac{1}{2}x^2 + Cx + D$$

$$\left\{ \begin{array}{l} U''(0) = \Theta(0-1) + A \cdot 0 + B = \Theta(-1) + B = 0 \Rightarrow B = 0 \\ U''(2) = \Theta(2-1) + 2A + B = \Theta(1) + 2A + 0 = 0 \Rightarrow A = -\frac{1}{2} \end{array} \right.$$

$$U(0) = \frac{1}{2}(0-1)^2\Theta(0-1) - \frac{1}{12} \cdot 0^3 + 0 + C \cdot 0 + D = 0 \Rightarrow D = 0$$

$$U(2) = \frac{1}{2}(2-1)^2\Theta(2-1) - \frac{1}{12} \cdot 2^3 + 0 + 2C + 0 = 0$$

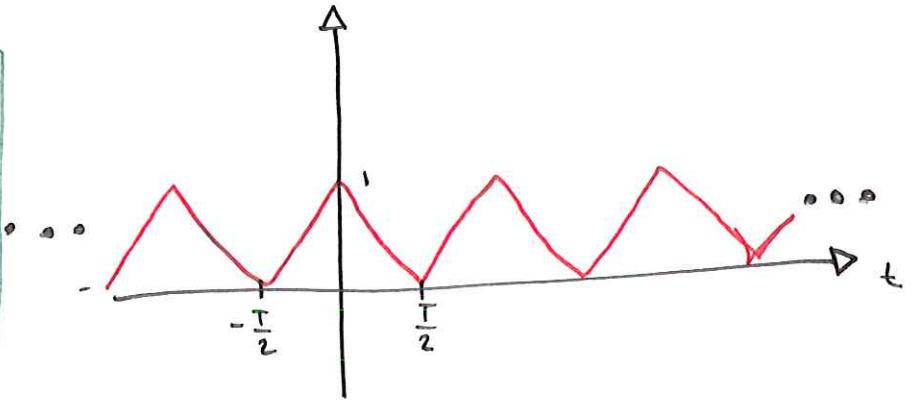
$$\Leftrightarrow \frac{1}{2} - \frac{8}{12} + 2C = 0 \Rightarrow C = \frac{1}{12}$$

$$\Rightarrow U(x) = \frac{1}{2}(x-1)^2\Theta(x-1) - \frac{1}{12}(x^3 - x)$$

11.22

Bestäm f' och f'' ;
distributionsmening

Bestäm fourierserien
för f



$f(t)$ har inga spräng

$$\Rightarrow f'(t) = \begin{cases} 2/T & (-T/2, 0) \\ -2/T & (0, T/2) \end{cases}, \quad f'' = 0 + \sum_{k=-\infty}^{\infty} \frac{4}{T} (-1)^{k+1} \delta\left(t - \frac{kT}{2}\right)$$

↑
Upprepas periodiskt

Diracs stakret

↑
Punktuvisa derivator

$$c_k(f'') = \frac{4}{T} (-1)^{k+1}$$

$$c_k(f') = \frac{4(-1)^{k+1}}{ik\Omega t} \Rightarrow c_k(f) = \frac{4(-1)^k}{k^2 \Omega^2 T}$$

$$\Rightarrow f = \sum_{k=-\infty}^{\infty} \frac{(-1)^k c_1}{k^2 \Omega^2 T} e^{ik\Omega t} + c_0, \quad c_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2}$$

$$\Rightarrow f = \sum_{k=-\infty}^{\infty} \frac{(-1)^k 4}{k^2 \Omega^2 T} e^{ik\Omega t} + \frac{1}{2}$$

11.25

Beräkna fältningen $f * g$ av distributionerna

a) $f(t) = \delta(t)$, $g(t) = \cos(2t)$

$$f * g = g * \delta(t) = \int_{-\infty}^{\infty} g(t - \tau) \delta(\tau) d\tau =$$

$$= [g(t - \tau)]_{\tau=0}^{\text{SVAR}} = g(t) = \cos 2t$$

b) $f(t) = \delta'(t)$, $g(t) = \cos 2t$

För varje distribution U gäller att

$$\delta' * U = U'$$

t_y

$$U' = \frac{d}{dt} U = \frac{d}{dt} (\delta * U) = \delta' * U$$

$$\Rightarrow \delta' * \cos 2t = -2 \sin 2t$$

c)

$$f(t) = \delta(t-1), g(t) = \cos 2t$$

$$f * g = \int_{-\infty}^{\infty} g(t-\tau) \delta(t-1) =$$

$$= \left[g(t-\tau) \right]_{\tau=1} = g(t-1) = \cos(2(t-1)) =$$

$$= \boxed{\cos(2t-2)}$$

d)

$$f(t) = \delta'(t), g(t) = \cos(2t) \oplus \Theta(t)$$

$$g(t) * \delta'(t) = g'(t) = -2 \sin(2t) \Theta(t) + \cos 2t \delta(t) =$$

$$= \boxed{-2 \sin 2t \Theta(t) + \delta(t)}$$

11.26

Beräkna $f * g$

$$f(t) = \delta''(t) - \delta(t) , \quad g(t) = e^{-t^2} + e^{-t} \cdot \Theta(t)$$

$$f * g = (\delta'' - \delta) * (e^{-t^2} + e^{-t} \Theta(t)) =$$

$$= \delta'' * (e^{-t^2} + e^{-t} \Theta(t)) - \delta * (e^{-t^2} + e^{-t} \Theta(t)) =$$

$$= \delta'' * e^{-t^2} + e^{-t} + \delta'' * e^{-t} \Theta(t) - (e^{-t^2} + e^{-t} \Theta(t)) =$$

$$= \delta'(t) * \delta'(t) * e^{-t^2} - e^{-t^2} - \delta' * \delta * e^{-t} \Theta(t) - e^{-t} \Theta(t) =$$

$$= \delta' * (-2t \cdot e^{-t^2}) - e^{-t^2} + \delta' * (-e^{-t} \Theta(t) + e^{-t} \delta(t)) - e^{-t} \Theta(t) =$$

$$= 4t^2 e^{-t^2} - 2e^{-t^2} - e^{-t^2} + (e^{-t} \Theta(t) - \delta \cdot e^{-t} + \delta' e^{-t}) - e^{-t} \Theta(t) =$$

$$= (4t^2 - 3) e^{-t^2} - \delta + \delta'$$

11.27

Bestim f * g

$$\cancel{f = \delta''(x) - \delta(x)}$$

$$\begin{cases} f = |x| = -x + 2x \Theta(x) = x(2\Theta(x) - 1) \\ g = \delta''(x) \end{cases}$$

$$f * g = x(2\Theta(x) - 1) * \delta''(x) = \frac{d^2}{dx^2}(x(2\Theta(x) - 1)) * \delta(x)$$

$$f' = 2\Theta(x) - 1 + x \cancel{(2\delta(x))}$$

$$f'' = 2\delta(x)$$

$$f * g = 2\delta(x) * \delta(x) = \boxed{2\delta(x)}$$