10.1 a) $\begin{vmatrix} \lambda+2 & -1 \\ -1 & \lambda+2 \end{vmatrix} = (\lambda+2)^2-1 = 0 \Leftrightarrow \lambda_1=-1,\ \lambda_2=-3$

$\lambda_1=-1$

$\begin{cases} x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = t \\ x_2 = t \end{cases} ; \quad v_1 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad t \neq 0.$

$\lambda_2=-3$

$\begin{cases} -x_1 - x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = t \\ x_2 = -t \end{cases} ; \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b) $\begin{vmatrix} \lambda-2 & 1 \\ -1 & \lambda \end{vmatrix} = \lambda(\lambda-2)+1 = \lambda^2-2\lambda+1 = (\lambda-1)^2 = 0 \Leftrightarrow \lambda=1$

$\lambda=1$

$\begin{cases} -x_1 + x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = t \\ x_2 = t \end{cases} ; \quad v = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c) $\begin{vmatrix} \lambda-1 & -3 \\ -12 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 36 = 0 \Leftrightarrow \lambda_1=-5,\ \lambda_2=7$

$\lambda_1=-5$

$\begin{cases} -6x_1 - 3x_2 = 0 \\ -12x_1 - 6x_2 = 0 \end{cases} ; \quad v_1 = t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2=7$

$\begin{cases} 6x_1 - 3x_2 = 0 \\ -12x_1 + 6x_2 = 0 \end{cases} ; \quad v_2 = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

10.1 d) $\begin{vmatrix} \lambda-1 & -5 \\ -7 & \lambda+1 \end{vmatrix} = (\lambda-1)(\lambda+1) - 35 = \lambda^2 - 36 = 0 \iff \lambda = \pm 6$

$\lambda_1 = 6$

$\begin{cases} 5x_1 - 5x_2 = 0 \\ -7x_1 + 7x_2 = 0 \end{cases}$; $v_1 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_1 = -6$

$\begin{cases} -7x_1 - 5x_2 = 0 \\ -7x_1 - 5x_2 = 0 \end{cases}$; $v_2 = t\begin{pmatrix} 5 \\ -7 \end{pmatrix}$

e) $\begin{vmatrix} \lambda-5 & -4 \\ -3 & \lambda-1 \end{vmatrix} = (\lambda-5)(\lambda-1) - 12 = \lambda^2 - 6\lambda - 7 = 0 \iff \lambda_1 = -1, \lambda_2 = 7$

$\lambda_1 = -1$; $v_1 = t \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$\lambda_2 = 7$; $v_2 = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

f) $\begin{vmatrix} \lambda-37 & -9 \\ 16 & \lambda-13 \end{vmatrix} = (\lambda-37)(\lambda-13) + 144 = \lambda^2 - 50\lambda + 625 = 0 \iff \lambda = 25$

$\lambda = 25$ ger

$\begin{cases} -12x_1 - 9x_2 = 0 \\ 16x_1 + 12x_2 = 0 \end{cases}$; $v = t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

10.2 b) $\left(\frac{1}{6}\right)^3 \begin{vmatrix} 6\lambda-5 & -1 & 2 \\ -1 & 6\lambda-5 & -2 \\ 2 & -2 & 6\lambda-2 \end{vmatrix} = \left(\frac{1}{6}\right)^3 \Big((6\lambda-5)^2 \cdot (6\lambda-2) + 4 + 4$

$- 4(6\lambda-5) - 4(6\lambda-5) - (6\lambda-2) \Big) = \left(\frac{1}{6}\right)^3 \Big((36\lambda^2 - 60\lambda + 25)(6\lambda-2) + 8$

$- 8(6\lambda-5) - 6\lambda + 2 \Big) = \left(\frac{1}{6}\right)^3 \Big(216\lambda^3 - 72\lambda^2 - 360\lambda^2 + 120\lambda + 150\lambda - 50$

$+ 8 - 48\lambda + 40 + 6\lambda + 2 \Big) = \left(\frac{1}{6}\right)^3 \cdot \left(216\lambda^3 - 432\lambda^2 + 216\lambda \right)$

$= \lambda^3 - 2\lambda^2 + \lambda = 0 \iff \lambda_1 = 0 \quad \lambda_2 = 1$

$\lambda_1 = 0$:
$$\begin{cases} -5x_1 - x_2 + 2x_3 = 0 \\ -x_1 - 5x_2 - 2x_3 = 0 \\ 2x_1 - 2x_2 - 2x_3 = 0 \end{cases} \overset{I \cdot 5}{\iff} \begin{cases} -x_1 - 5\overset{-t}{x_2} - 2\overset{2t}{x_3} = 0 \\ 24x_2 + 12x_3 = 0 \\ -12x_2 - 6x_3 = 0 \end{cases} \iff \begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = 2t \end{cases}$$

$$v_1 = t \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$\lambda_2 = 1$
$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ -x_1 + x_2 - 2x_3 = 0 \\ 2x_1 - 2x_2 + 4x_3 = 0 \end{cases} \iff \begin{cases} x_1 = s - 2t \\ x_2 = s \\ x_3 = t \end{cases}$$

$$v_2 = t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

10.2 d) $\begin{vmatrix} \lambda-1 & -1 & -1 \\ -2 & \lambda-1 & 1 \\ -2 & -3 & \lambda-5 \end{vmatrix} = (\lambda-1)^2(\lambda-5)+2-6-2(\lambda-1)+3(\lambda-1)-2(\lambda-5)$

$= (\lambda^2-2\lambda+1)(\lambda-5)-4+\lambda-1-2\lambda+10 =$

$= \lambda^3-5\lambda^2-2\lambda^2+10\lambda+\lambda-5+5-\lambda =$

$= \lambda^3-7\lambda^2+10\lambda = \lambda(\lambda^2-7\lambda+10) = 0$

$\Leftrightarrow$

$\lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = 5$

$\lambda_1 = 0;$ $\begin{cases} -x_1 - x_2 - x_3 = 0 \\ -2x_1 - x_2 + x_3 = 0 \\ -2x_1 - 3x_2 - 5x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} -2x_1 - 3x_2 - 5x_3 = 0 \\ x_2 + 3x_3 = 0 \\ -x_2 - 3x_3 = 0 \end{cases}$ $; \; v_1 = t\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$\lambda_1 = 2$ $\begin{cases} x_1 - x_2 - x_3 = 0 \\ -2x_1 + x_2 + x_3 = 0 \\ -2x_1 - 3x_2 - 3x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 - x_2 - x_3 = 0 \\ -x_2 - x_3 = 0 \\ -5x_2 - 5x_3 = 0 \end{cases}$ $; \; v_2 = t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\lambda_3 = 5$ $\begin{cases} 4x_1 - x_2 - x_3 = 0 \\ -2x_1 + 4x_2 + x_3 = 0 \\ -2x_1 - 3x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} 4x_1 - x_2 - x_3 = 0 \\ 2x_1 + 3x_2 = 0 \\ -2x_1 - 3x_2 = 0 \end{cases}$ $; \; v_3 = t\begin{pmatrix} -3 \\ 2 \\ -14 \end{pmatrix}$

10.2e) $\begin{vmatrix} \lambda-1 & -2 & 0 \\ -1 & \lambda-1 & -2 \\ 0 & 1 & \lambda-1 \end{vmatrix} = (\lambda-1)^3 + 2(\lambda-1) - 2(\lambda-1) = 0 \Leftrightarrow \lambda = 1$

$\lambda=1$: $\begin{cases} -2x_2 = 0 \\ -x_1 \quad -2x_3 = 0 \\ x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -2t \\ x_2 = 0 \\ x_3 = t \end{cases}$; $v = t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

10.2 f) $\begin{vmatrix} \lambda-1 & -2 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 1 & \lambda-1 \end{vmatrix} = (\lambda-1)^3 = 0 \Leftrightarrow \lambda = 1$

$\lambda=1$

$\begin{cases} -2x_2 = 0 \\ 0 = 0 \\ x_2 = 0 \end{cases}$ $v = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

10.3) Projektion i planet $x_1 - x_2 + 2x_3 = 0$ i riktningen $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Detta "råkar" vara ortogonal projektion ty planets normal är just $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

10.4) $\begin{vmatrix} \lambda-1 & 2 & -2 \\ 2 & \lambda-4 & 4 \\ -2 & 4 & \lambda-4 \end{vmatrix} = (\lambda-1)(\lambda-4)^2 - 16 - 16 - 4(\lambda-4) - 16(\lambda-1) - 4(\lambda-4) =$

$= (\lambda-1)(\lambda^2-8\lambda+16) - 32 - 8(\lambda-4) - 16\lambda + 16 =$

$= \lambda^3 - 8\lambda^2 + 16\lambda - \lambda^2 + 8\lambda - 16 - 32 - 8\lambda + 32 - 16\lambda + 16 =$

$= \lambda^3 - 9\lambda^2 = 0 \quad \Leftrightarrow \quad \lambda_1 = 0 \;, \lambda_2 = 9$

$\lambda = 0$

$$\begin{cases} -x_1 + 2x_2 - 2x_3 = 0 \\ 2x_1 - 4x_2 + 4x_3 = 0 \\ -2x_1 + 4x_2 - 4x_3 = 0 \end{cases} \quad \Leftrightarrow \quad x_1 - 2x_2 + 2x_3 = 0$$

$\lambda = 9$

$$\begin{cases} 8x_1 + 2x_2 - 2x_3 = 0 \\ 2x_1 + 5x_2 + 4x_3 = 0 \\ -2x_1 + 4x_2 + 5x_3 = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 18x_2 + 18x_3 = 0 \\ 9x_2 + 9x_3 = 0 \\ \sim \end{cases} \quad \Big| \; v = t\begin{pmatrix} -1/2 \\ 1 \\ -1 \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

normalriktning till planet $x_1 - 2x_2 + 2x_3$

Matrisen projicerar ortogonalt i $x_1 - 2x_2 + 2x_3 = 0$ och förlänger den projicerade vektorn med faktor 9.

10.6) $AX = \lambda X$ (vet)

$BX = (A^3 - 5A^2 + A + 7I)X = A^3X - 5A^2X + AX + 7X =$

$= \lambda^3 X - 5\lambda^2 X + \lambda X + 7X = \underbrace{(\lambda^3 - 5\lambda^2 + \lambda + 7)}_{\substack{\gamma \\ \text{egenvärdet.}}} X = \gamma X$

10.7 $\longrightarrow$ D

10.8) a)
$$\begin{vmatrix} \lambda-1 & -7 & -3 \\ 0 & \lambda-2 & -5 \\ 0 & 0 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$

b) $\lambda_1 = a_{11}, \ldots, \lambda_n = a_{nn}$

"Egenvärdena = diagonalelementen"

10.10 a) $D = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix},\ S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

b) Nix!

c) $D = \begin{pmatrix} -5 & 0 \\ 0 & 7 \end{pmatrix},\ S = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$

d) $D = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix},\ S = \begin{pmatrix} 1 & 5 \\ 1 & -7 \end{pmatrix}$

e) $D = \begin{pmatrix} -1 & 0 \\ 0 & 7 \end{pmatrix},\ S = \begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix}$

f) Nix!

10.7) a) $\lambda_1 = 1$ $v_1 = (1, 2, 3)$

$\lambda_2 = 0$ $v_2 = t(2, -1, 0) + s(0, 3, -2)$

b) $\lambda_1 = 1$ v_1 i planet $x_1 + 2x_2 + 3x_3 = 0$

$\lambda_2 = 0$ $v_2 = t(1, 2, 3)$

c) $\lambda_1 = 1$ v_1 i planet

$\lambda_2 = -1$ $v_2 = t(1, 2, 3)$

10.11) a) Sparr sid 238

b) $$\begin{vmatrix} \lambda-3 & 2 & -2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda-3 \end{vmatrix} = (\lambda-3)^2\cdot\lambda+4+2-2\lambda+2(\lambda-3)+2(\lambda-3)=$$

$$= \lambda^3-6\lambda^2+9\lambda+6-2\lambda+2\lambda-6+2\lambda-6=$$

$$= \lambda^3-6\lambda^2+11\lambda-6=(\lambda-1)(\lambda^2-5\lambda+6)=0$$

$$\Leftrightarrow \quad \lambda_1=1,\ \lambda_2=2,\ \lambda_3=3$$

$\lambda_1=1$

$$\begin{cases} -2x_1+2x_2-2x_3=0 \\ -x_1+x_2-2x_3=0 \\ -x_1+x_2-2x_3=0 \end{cases} \Leftrightarrow \begin{cases} 2x_3=0 \\ x_1+x_2-2x_3=0 \\ 0=0 \end{cases} ; \quad v_1=t\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$\lambda_2=2$

$$\begin{cases} -x_1+2x_2-2x_3=0 \\ -x_1+2x_2-2x_3=0 \\ -x_1+x_2-x_3=0 \end{cases} \Leftrightarrow \begin{cases} -x_1+2x_2-2x_3=0 \\ 0=0 \\ -x_2+x_3=0 \end{cases} ; \quad v_2=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$\lambda_3=3$

$$\begin{cases} 2x_2-2x_3=0 \\ -x_1+3x_2-2x_3=0 \\ -x_1+x_2 \quad =0 \end{cases} \Leftrightarrow \begin{cases} 2x_2-2x_3=0 \\ 2x_2-2x_3=0 \\ -x_1+x_2 \quad =0 \end{cases} ; \quad v_3=t\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$D=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}, \quad S=\begin{pmatrix}1&0&1\\1&1&1\\0&1&1\end{pmatrix}$$

10.13) $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

$$\begin{vmatrix} \lambda+2 & -1 \\ -1 & \lambda+2 \end{vmatrix} = (\lambda+2)^2 - 1 = 0 \Leftrightarrow \lambda_1 = -1,\ \lambda_2 = -3$$

$\lambda_1 = -1$; $\begin{cases} x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases}$; $v_1 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -3$; $\begin{cases} -x_1 - x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases}$; $v_2 = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$D = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$A = S^{-1}$

$A^{10} = (SDS^{-1})^{10} = SD^{10}S^{-1} =$

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^{10} \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$

$= \begin{pmatrix} 1 & 3^{10} \\ 1 & -3^{10} \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+3^{10} & 1-3^{10} \\ 1-3^{10} & 1+3^{10} \end{pmatrix}$

$Y = AX$

"$v = S^T e$"

$X = SX'$

$Y = SY'$

$SY' = ASX'$

$Y' = \underbrace{S^{-1}AS}_{D}X'$

$SDS^{-1} = A$

10.15) $A = SIS^{-1} = I$.

10.22) $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ $A = A^T$ så A symmetrisk och har därmed ON-bas av egenvektorer.

a) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} = -2\begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$\lambda_1 = -2, \quad v_1 = t(-2, 2)$ Ja

Alltså måste $V_2 = t(1, 1)$ vara egenvektor

$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 4, \quad v_2 = t(1, 1)$

c) $S = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

Facit $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$?!

10.27) A har egenvärden $0, 1, 2$. Visa $I+A$ inv. bar.
3×3

$I+A$ har alltså e.v. $1, 2, 3$ ty

$$(A+I)v = Av + Iv = \lambda v + v = (\lambda+1)v$$

Således finns S så att

$$S^{-1}(A+I)S = \underset{\begin{pmatrix} 1 & & 0 \\ & 2 & \\ 0 & & 3 \end{pmatrix}}{D}$$

Som D inv bar så

$$(A+I)^{-1} = (SDS^{-1})^{-1} = SD^{-1}S^{-1}$$