

Summary of Chapter 1 slides - Numdiff

Initial value problems 9/12-2015

Lipschitz condition

$$\|f(t, u) - f(t, v)\| \leq L[f] \cdot \|u - v\|$$

↖ Lipschitz constant

If $L[f] < \infty$, then there exists a unique solution to the initial value problem on $[0, T]$ for every initial value $y(0) = y_0$.

The matrix norm $\|A\|$ is a Lipschitz constant for the equation $f(y) = Ay$

$$\begin{aligned} \text{Ex: } \dot{y} = Ay \Rightarrow L[f] &= \max_{u \neq v} \frac{\|Au - Av\|}{\|u - v\|} = \|y = u - v\| = \cancel{\|y\|} \\ &= \max_{y \neq 0} \frac{\|Ay\|}{\|y\|} = \boxed{\|A\|} \end{aligned}$$

Standard form

$$y'' = f, \quad y(0) = y_0, \quad y'(0) = y'_0$$

$$\text{Substitution: } \begin{cases} x'_1 = x_2 \\ x'_2 = f \end{cases}, \quad x_1(0) = y_0, \quad x_2(0) = y'_0$$

How should f be defined for this to work?

$$f(t, y, y') \Rightarrow \text{GREAT}$$

$$f(t, y) \Rightarrow \text{PROBLEMATIC}$$

The explicit Euler method,

$$\begin{cases} Y_{n+1} = Y_n + h \cdot f(t_n, Y_n) \\ t_{n+1} = t_n + h \end{cases}$$

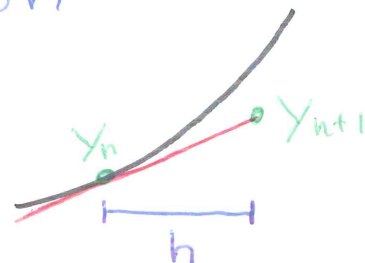
solution

$$y' = f(t, y)$$

Equation

Taylor: $y(t+h) = y(t) + h y'(t) + O(h^2)$

$\nearrow f(t, y)$



Local error

Error caused by one step.

Global error

Total error caused by multiple steps.

Convergent: $\lim_{N \rightarrow \infty} \|Y_{N,h} - y(T)\| = 0$ for every $T = N \cdot h$.

Is explicit Euler convergent?

Lemma: assume $a_{n+1} \leq (1 + h\mu) a_n + ch^2$
then $a_n \leq \frac{ch}{\mu} ((1 + h\mu)^n - 1)$
 $\uparrow$ interest $\uparrow$ deposit

Apply the lemma to global error recursion, to get:

$$\|\text{error}_{n,h}\| \leq \frac{c}{L[f]} \cdot h [(1 + hL[f])^n - 1] = C(T) \cdot h \rightarrow \underline{\underline{0}}, \quad h \rightarrow 0$$

THE ERROR GOES TO ZERO ~~AND~~
WHEN THE STEPSIZE GOES TO ZERO!

Theoretical error bound:

$$\text{Eq: } y' = -100y, \quad y(0) = 1$$

$$L[f] \geq \frac{\| -100u - (-100v) \|}{\| u - v \|} = \frac{\| -100y \|}{\| y \|} = \boxed{100} \quad L[f]$$

$$\text{exact solution: } \boxed{y(t) = e^{-100t}} \quad \text{exact sol.}$$

$$\begin{aligned} \|\text{error}\| &\leq \frac{c}{L[f]} \cdot h \left((1 + hL[f])^n - 1 \right) = \\ &= \frac{100^2/2}{100} \cdot h \left(e^{100\tau} - 1 \right) \stackrel{\tau=1}{\leq} 5e^{100} \cdot h \approx \boxed{1.4 \cdot 10^{45} h} \end{aligned}$$

A LOT

$$\text{Actual error: } y_n = (1 - 100h)^n, \quad \tau = 1, \quad h \leq 1/50$$

$$\Rightarrow \|\text{actual error}\| = \left\| \left(1 - \frac{100}{N} \right)^N - e^{-100} \right\| \leq \boxed{3.7 \cdot 10^{-44}}$$

THE ERROR IS OVERESTIMATED BY
AT LEAST 89 ORDERS OF MAGNITUDE.

$\Rightarrow$ This is why "real" math sucks. // Numdiffers

~~Order of convergence~~
^{sistency}

The ~~OC~~ (order of consistency) p is given in

The order of Consistency is p if

$$Y(t_{n+1}) - \Phi_h(f, h, Y(t_n), Y(t_{n+1}), \dots) = O(h^{p+1})$$

Just insert monomes to test this.

The linear test equation

$$\text{Eq: } Y' = \lambda Y, Y(0) = 1, t \geq 0, \lambda \in \mathbb{C}$$

Bounded sols if $\text{Re}(\lambda) \leq 0$

DOES NOT IMPLY NUMERICAL STABILITY!

$$\text{DEF: } |Y_n| \leq \tilde{K}, \text{ ex: } Y_{n+1} = (1+h\lambda)Y_n \Rightarrow |1+h\lambda| \leq 1$$

small $h \Rightarrow$ stability

Numerical instability increases exponentially and have oscillatory behaviour!

DEF: A-stability

Stiff equations

"If explicit solvers need very small stepsize to solve the eq., then the eq. is stiff."

Homogenous sols are very damped.

Implicit methods with unbounded stability regions put no stability restrictions on h .