

# SUMMARY OF CHAPTER 5 SLIDES - NUMDIFF

## Elliptic and Parabolic PDEs

10/12-2015

### CLASSIFICATION

Poisson eq:  $-\Delta u = f + BC$  ELLIPTIC

Diffusion eq:  $u_t = \Delta u + BC \& IV$  PARABOLIC

Wave eq:  $u_{tt} = \Delta u + BC \& IV$  HYPERBOLIC

Advection eq:  $u_t + a(u) \cdot \nabla u = 0 + BC \& IV$  HYPERBOLIC

### Classical approach

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + L(u_x, u_y, u, x, y) = 0$$

DEF:  $\Delta = \det \begin{pmatrix} A & B \\ B & C \end{pmatrix} = AC - B^2$

$\Delta > 0 \Rightarrow$  Elliptic

$\Delta = 0 \Rightarrow$  Parabolic

$\Delta < 0 \Rightarrow$  Hyperbolic

How can we try this out?

EXAMPLE:

Hyperbolic if  $p+q$  is even.  
Parabolic if  $p+q$  is odd.

Highest real order terms!

$$\frac{\partial^p u}{\partial t^p} = \frac{\partial^q u}{\partial x^q}$$

$$u_t = u_x$$

Hyperbolic

$$u_t = u_{xx}$$

Parabolic

$$u_t = u_{xx}$$

Hyperbolic

↑ not real!

$$u_t = u_{xxx}$$

Hyperbolic

$$u_t = -u_{xxxx}$$

Parabolic

$$u_{tt} = u_{xx}$$

Hyperbolic

$$u_{tt} = -u_{xxxx}$$

Hyperbolic

Strong form:  $-\Delta u = f$ ;  $u=0$  on  $\partial\Omega$

Irreversability:  $u_t = -\Delta u$  is not well posed  
 $\Rightarrow$  We cannot run diffusion problems in backward time.

### Method of lines (MOL)

In  $u_t = u_{xx}$ , discretize  $\frac{\partial^2}{\partial x^2}$  by:

$$u_{xx} \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2}$$

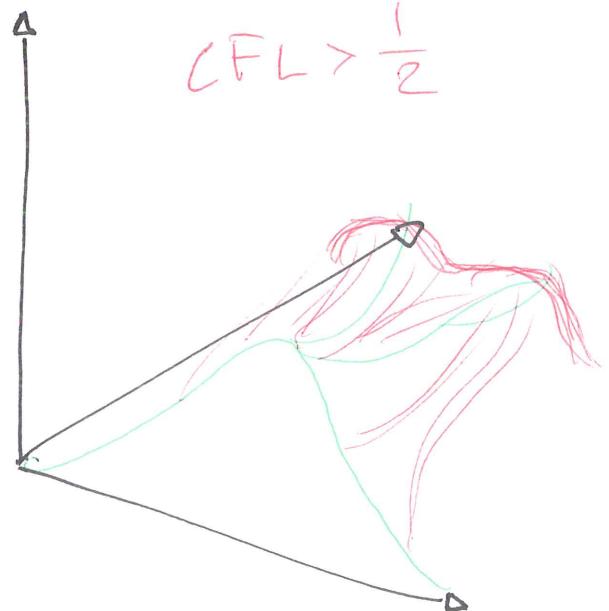
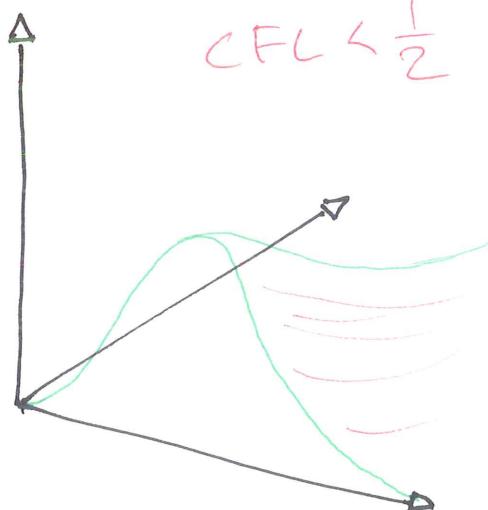
Note:  $u_i(t) \approx u(t, x_i)$  along the line  $x=x_i$  in  $(t, x)$  plane.

Courant number:  $\mu = \frac{\Delta t}{\Delta x^2}$

CFL-condition:

$$\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

Important!



## Crank-Nicolson method (solves $U_t = \alpha U_{xx}$ )

Same as the trapezoidal rule for PDEs.

There is no CFL condition on the time-step  $\Delta t$  which is why the Crank-Nicolson method is preferable.

$$\text{Th: } 2[A_\mu] = \frac{1 + \frac{\mu}{2} \lambda[T]}{1 - \frac{\mu}{2} \lambda[T]}$$

$$\mu = \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

For every explicit method, there is a CFL-condition.

## The convection-diffusion equation

$$U_t = U_{xx} + \alpha U_x - f$$

convection = transport.

Diffusion = concentration loss.

## Well-posedness

Suppose  $u(x,t) = \mathcal{E}(t) \bullet g(x)$

DEF: The eq. is well-posed for every  $t^* > 0$  if there is a constant  $0 < C(t^*) < \infty$  such that  $\|\mathcal{E}(t)\| \leq C(t^*)$  for all  $0 \leq t \leq t^*$ .

A well posed eq. has a solution that:

- Depends continuously on the initial value.
- Is uniformly bounded in any compact interval.

## Do this:

\* Prove that  $U_t = U_{xx}$  is well posed!