

SUMMARY OF CHAPTER 4 SLIDES - NUMDIFF

10/12-2015

FROM FINITE DIFFERENCES TO FINITE ELEMENTS

Inner product: $\langle u, v \rangle = \int_0^1 u \cdot v \, dx \Rightarrow \|u\|_2^2 = \langle u, u \rangle$

Can we find a constant such that:

$$\langle u, u'' \rangle \leq M_2 \left[\frac{d^2}{dx^2} \right] \cdot \|u\|_2^2$$

YES, $M_2 \left[\frac{d^2}{dx^2} \right] = -\pi^2$

Awesome formula

$$\langle u, v' \rangle = -\langle u', v \rangle$$

Because $u(0) = u(1)$

$$\langle u, v' \rangle = \int_0^1 uv' \, dx = [uv]_0^1 - \int_0^1 u'v \, dx = -\langle u', v \rangle$$

Sobolov's lemma

For all functions u with $u(0) = u(1) = 0$ it holds that

$$\|u'\|_2 \geq \pi \|u\|_2$$

Proof with Parseval's theorem:

$$u = \sqrt{2} \sum_{k=1}^{\infty} c_k \sin k\pi x \Rightarrow u' = \sqrt{2} \sum_{k=1}^{\infty} k\pi c_k \cos(k\pi x) \quad \square$$

Equality for $u(x) = \sin(\pi x)$.

$$Th: M_2 \left[\frac{d^2}{dx^2} \right] = -\pi^2$$

DEF: if $\langle v, Au \rangle = \langle A^*v, u \rangle$, then A^* is the adjoint operator.

DEF: self-adjoint $\Rightarrow A = A^*$.

All self-adjoint operators have real eigenvalues!

Anti self-adjoint op. ($A^* = -A$) have imaginary λ .

$\frac{d^2}{dx^2}$ - Self-adjoint , $\frac{d}{dx}$ - Anti self-adjoint

This means that diffusion problems have real λ and wave problems have imaginary λ .

Orthogonal eigenvectors: Let $Au = \lambda u$ & $Av = \mu v$

$$\Rightarrow \lambda \langle v, u \rangle = \langle v, Au \rangle = \langle A^*v, u \rangle = \langle Av, u \rangle = \mu \langle v, u \rangle$$

$$\lambda \neq \mu \Rightarrow \langle v, u \rangle = 0. \quad \square$$

Elliptic operators

expectation value of A is the same as λ . $\Rightarrow \lambda_k > 0$

DEF: $\langle u, Au \rangle > 0$, ex: $-\Delta$ is elliptic

DEF: An operator is positive definite if it is self-adjoint and elliptic

Finite difference method

- Replace u and f by vectors
- Replace A by a matrix
- Obtain linear system of equations

Weak formulation

$$\langle v', u' \rangle = \langle v, f \rangle \quad \forall v \quad (\text{several } f \text{ may satisfy this})$$

Energy norm

DEF: $a(v, u) = \langle v', u' \rangle$

The weak formulation can be written: $a(v, u) = \langle v, f \rangle$.

Finite element method

Choose ~~*~~ piecewise linear basis polynomials.

$$v(x) = \sum_{j=1}^n c_j \ell_j(x), \text{ Note: } v(x_i) = c_i \approx u(x_i)$$

Galerkin continuous G(1) method

Best approx: $a(v, u) = \langle v, f \rangle$ with $u, v \in V$

$$\Rightarrow a(u_i, \sum_{j=1}^n c_j \ell_j) = \langle \ell_i, f \rangle$$

which is equivalent to $Kc = b$.

~~Reaktionen~~

Stiffness matrix: $K_{\Delta x} = \frac{1}{\Delta x} \text{tridiag}(-1, 2, -1)$

Mass matrix: $B_{\Delta x} = \frac{\Delta x}{6} \text{tridiag}(1, 4, 1)$