Summary of Chapter 3 slides - Numdiff

10/12-2015

First order approximations:

- Forward difference: $y'(x) = \frac{y(x+\Delta x) - y(x)}{\Delta x} + O(\Delta x)$

- Backward difference: $y'(x) = \frac{y(x) - y(x-\Delta x)}{\Delta x} + O(\Delta x)$

Second order approximations:

- $y'(x) = \frac{y(x+\Delta x) - y(x-\Delta x)}{2\Delta x} + O(\Delta x^2)$

- $y''(x) = \frac{\frac{y_{n+1} - y_n}{\Delta x} - \frac{y_n - y_{n-1}}{\Delta x}}{\Delta x} = \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{\Delta x^2} + O(\Delta x^2)$

From derivates to matrices

- All operators can be represented by a matrix.

Example: forward difference: $\begin{pmatrix} y_0' \\ y_1' \\ \vdots \\ y_N' \end{pmatrix} \approx \frac{1}{\Delta x} \begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \\ & & & -1 \; 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{pmatrix}$

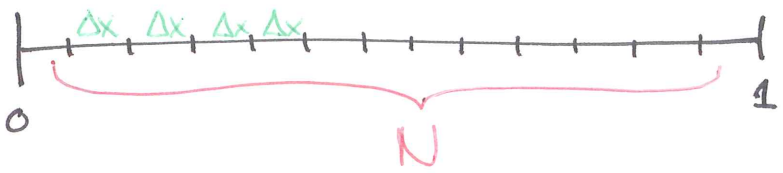
Nullspace: $y = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

DEFINITION

Nullspace: For which x is $Ax = 0$?

Finite difference methods for 2p-BVP.

Grid with stepsize $\Delta x = \frac{1}{N+1}$



To discretize, always write the following:

- $y'' = f(x,y)$, $y(0) = \alpha$, $y(1) = \beta$

ALWAYS

- $F_1(y) = \frac{\alpha - 2y_1 + y_2}{\Delta x^2} - f(x_1, y_1)$
- $F_i(y) = \frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} - f(x_i, y_i)$
- $F_N(y) = \frac{y_{N-1} - 2y_N + \beta}{\Delta x^2} - f(x_N, y_N)$

Boundry conditions

Dirchlet: $y(0) = \alpha$, $y(1) = \beta$

Neumann: $y'(0) = \gamma$, $y'(1) = \beta$

Robin: $y(0) + cy'(0) = \kappa$, $y(1) = \beta$

Remember: $\Delta x = \frac{L}{N+0,5}$ (Neumann) for Neumann problems!

Sturm-Liouville eigenvalue problem

DEF: $\frac{d}{dx}\left(p(x)\frac{d}{dx}y(x)\right) = \lambda y$

Example: $y'' = \lambda y \overset{y(0)=y(1)=0}{\Rightarrow} y = A\sin(\sqrt{-\lambda}x) + B\cos(\sqrt{-\lambda}x)$

$y(0) = 0 \Rightarrow B = 0$

$y(1) = 0 \Rightarrow A\sin(\sqrt{-\lambda}) = 0 \Rightarrow \lambda_k = -(k\pi)^2$, $k = 1, 2, \ldots$

$\Rightarrow \boxed{y_k(x) = A\sin(k\pi x)}$

Discretation of $y'' = \lambda y$ with BVs

Dirchlet!

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} = \lambda_{\Delta x} y_i \quad , \quad \boxed{y_0 = y_{N+1} = 0, \ \Delta x = \frac{1}{N+1}}$$

Toeplitz matrix

$$T_{\Delta x} = \frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \\ & & 1 & -2 \end{bmatrix} = \text{tridiag}(1, -2, 1)$$

$\Rightarrow \boxed{T_{\Delta x} y = \lambda_{\Delta x} y}$ MATLAB: $\lambda_{\Delta x} = \text{eig}(T_{\Delta x})$

Note: { Lowest λ more accurate
Good approximation for $\sqrt{N}$ eigenvalues.

DEF: Toeplitz matrix is constant along its diagonals.

Eigenvalues to Toeplitz matrices

$p(\lambda) = \det(T - \lambda I)$, assume $T = S - 2I$

$\Rightarrow \det(T - \lambda I) = \det(S - 2I - \lambda I) = \det(S - (\lambda + 2)I)$

Note $\boxed{\lambda[T] = -2 + \lambda[S]}$ Important

Now, find eigenvalues to $S = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix}$

Solve $Sy = \lambda y \iff y_{n+1} + y_{n-1} = \lambda y_n$

Remember funken: This is a linear diff. eq.

Char eq: $z^2 - \lambda z + 1 = 0$

Compare with: $(z-\alpha)(z-\beta) = z^2 - (\alpha+\beta)z + \alpha\beta$

$$\Rightarrow \begin{cases} \alpha\beta = 1 \\ \alpha+\beta = \lambda \end{cases}, \lambda \text{ is } \Rightarrow \text{Roots: } \begin{cases} z \\ 1/z \end{cases}$$

$$y_n = Az^n + Bz^{-n}$$

$$y_0 = 0 \Rightarrow y_n = A(z^n - z^{-n})$$

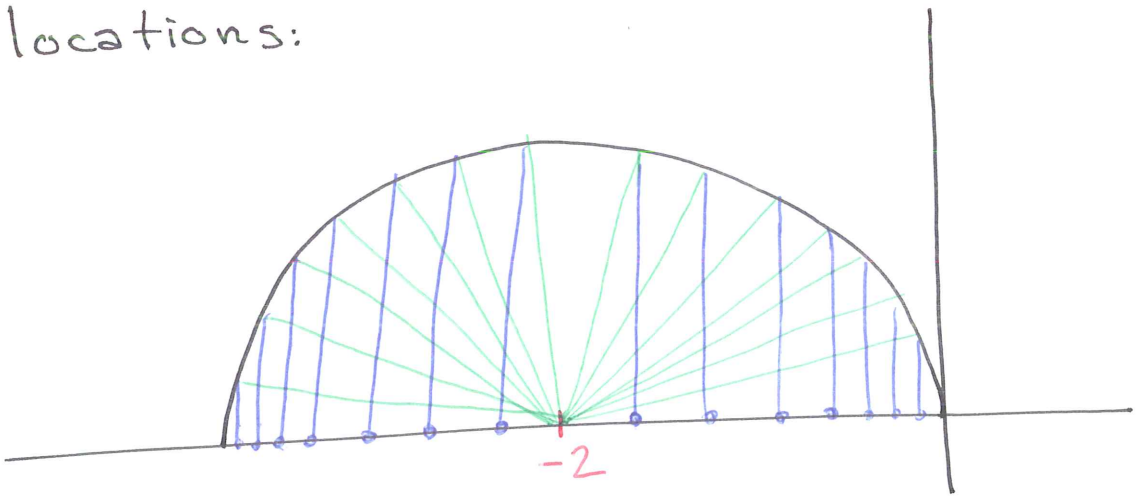
$$y_{N+1} = 0 = A(z^{N+1} - N^{-N-1})$$

$$\Rightarrow z^{2(N+1)} = 1 \Rightarrow z_k = e^{\frac{k\pi i}{N+1}}, \quad k = 1, \dots, N$$

$$\Rightarrow \lambda_k[S] = z_k + \frac{1}{z_k} = 2\cos\left(\frac{k\pi}{N+1}\right)$$

$$\Rightarrow \boxed{\lambda_k[T] = -4\sin^2\left(\frac{k\pi}{2(N+1)}\right)} \quad eig(T)$$

locations:



Norms

A symmetric $\Rightarrow \|A\|_2 = \max_k |\lambda_k|$

A symmetric $\Rightarrow \mu_2[A] = \max_k \lambda_k$

Logarithmic norm.

DEF: $\mu_2[A] = \max_{x^T x \neq 0} \left(\frac{x^T A x}{x^T x} \right)$

Th: $\|T_{\Delta x}\|_2 = \frac{4}{\Delta x^2}$, $\mu_2[T_{\Delta x}] = -1$

Root mean square norm (RMS):

$$\|u\|_{\Delta x}^2 = \sum_{i=1}^{N} u(x_i)^2 \Delta x = \frac{1}{N+1} \sum_{i=1}^{N} u(x_i)^2 = \frac{1}{N+1} \|u\|_2^2$$

$\|u\|_{\Delta x}^2$

RMS: $\|u\|_{\Delta x} = \sqrt{\frac{1}{N+1} \sum_{i=1}^{N} u(x_i)^2}$

square, Mean, Root

The Lax principle

Fundamental theorem of numerical analysis

Consistency + Stability $\Rightarrow$ Convergence

Local error $\to 0$

Bounded second derivative ($\|T_{\Delta x}^{-1}\|_2 \leq C$)

Global error $\to 0$