Seminarium 2015-09-18

① Rita kurvan $2x^2 - y^2 - 4(x-y) - 3 = 0$ (0,5p)

och ange asymptoter 0,5p

Få bort x/y termer

$2x^2 - 4x$

$\Downarrow$

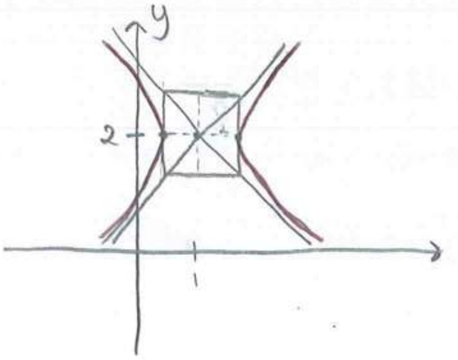
$2(x^2 - 2x + 1^2 - 1^2) - (y^2 - 4y + 2^2 - 2^2) - 3 = 0$

$2((x-1)^2 - 1) - ((y-2)^2 - 4) = 3$

$2(x-1)^2 - 2 - (y-2)^2 + 4 = 3 \iff 2(x-1)^2 - (y-2)^2 = 1$

$\dfrac{(x-1)^2}{\left(\sqrt{\frac{1}{2}}\right)^2} - \dfrac{(y-1)^2}{1^2} = 1$ Hyperbel

$a = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} \quad b = 1$



asymptoter

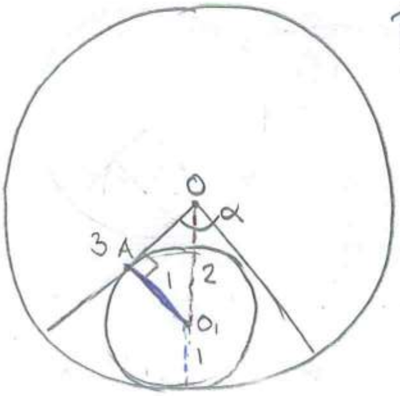
$\dfrac{x}{a} \pm \dfrac{y}{b} = 0$ dvs. $\dfrac{x-1}{\sqrt{\frac{1}{2}}} + \dfrac{y-2}{1} = 0$

och

$\dfrac{x-1}{\sqrt{\frac{1}{\sqrt{2}}}} - \dfrac{y-2}{1} = 0$

$\sqrt{2}\,(x-1) - (y-2) = 0$

② Bestäm α (0,4 p)



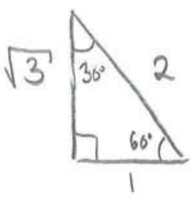
$\alpha = 30° \cdot 2 = 60°$

$2 = 3 - 1$

$\angle OAO_1 = 90°$

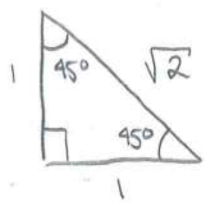
$OO_1 = 3 - 1 = 2$

$\sin \dfrac{\alpha}{2} = \dfrac{1}{2}$



$\sin 30° = \dfrac{1}{2}$

$\tan 60° = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

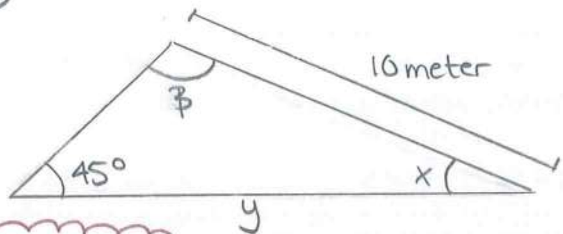


$\cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

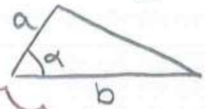
Man får använda triangeln på tenta för att bevisa svar.

③

(0,5p)



Area $\Delta = \frac{1}{2} ab \cdot \sin\alpha$



$\beta = 180 - 45° - x = 135° - x$

Sinussatsen ger

$$\frac{y}{\sin\beta} = \frac{10}{\sin 45°} \iff y = \frac{10}{\sin 45°} \cdot \sin\beta = 10\sqrt{2}\sin(135° - x)$$

$\uparrow \frac{1}{\sqrt{2}}$

Gör samma sak med a och sätt in i formel för Δ area.

④ Lös $4\lg\sqrt{x} + \lg((x-3)^2) = 2$ (0,4p)

Lösning: $4\lg\sqrt{x} + \lg((x-3)^2) = 2 \iff \lg(\sqrt{x})^4 + \lg((x-3)^2) = 2$

$$\iff 10^{\lg x^2 + \lg((x-3)^2)} = 10^2$$

$$\iff 10^{\lg x^2} \cdot 10^{\lg((x-3)^2)} = 100$$

$$\Rightarrow x^2 (x-3)^2 = 100 \iff (x(x-3))^2 = 100$$

$$\iff x(x-3) \pm 10$$

dvs. $x^2 - 3x - 10 = 0$ eller $\underbrace{x^2 - 3x + 10 = 0}_{\text{saknar lösning}}$

$x = -2$ $x = 5$ ok, från verifiering!

↑ falsk

⑤ $x + x^2 + x^3 + x^4 + x^5 + x^6 < 0$ (0,5p)

Lösning: $x(1 + x + x^2) + x^4(1 + x + x^2) < 0$

$x(1 + x + x^2)(1 + x^3) < 0$

$$\begin{array}{l} \quad x^2 - x + 1 \\ \hline x^3 + 1 \quad \big| \, x + 1 \\ x^3 + x^2 \\ \hline -x^2 + 1 \\ -x^2 - x \\ \hline x + 1 \\ x + 1 \\ \hline 0 \end{array}$$

$x^3 + 1 = (x+1)(x^2 - x + 1)$

Så har vi olikheten:

$x(1 + x + x^2)(1 + x)(x^2 - x + 1) < 0$

Låt $x = 0 \Rightarrow x = 0$
$1 + x = 0 \Rightarrow x = -1$

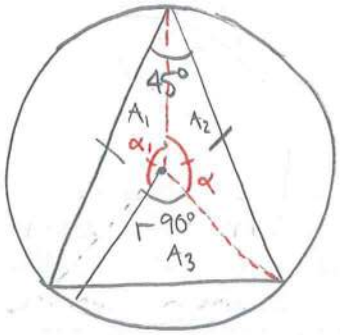
x		-1		0	
f(x)	+	0	−	0	+

Lösningar

$-1 < x < 0$

⑥

P35



$r = 2$ Lösning

Arean av triangeln

$\alpha_1 + \alpha + 90° = 360°$

$\alpha = 135°$

$\alpha_1 = \alpha$ då triangeln är likbent.

$A_1 = A_2 = \frac{1}{2} \cdot 2 \cdot 2 \cdot \sin 135° = 2 \sin(90° + 45°) = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$

$A_3 = \frac{1}{2} \cdot 2 \cdot 2 = 2$

Arean: $\sqrt{2} + \sqrt{2} + 2 = 2\sqrt{2} + 2$

⑥ Bestäm inversen till $f(x) = x^2 + 4x + 5 \quad x \geq -2$

$y = x^2 + 4x + 5$

$x^2 + 4x + 2^2 \quad + 1 - y = 0$

$(x+2)^2 = y - 1$

$x + 2 = \cancel{\pm} \sqrt{y-1}$

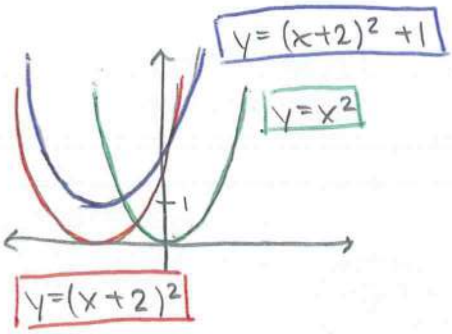
↑ ty $x \geq -2$

$x = 2 + \sqrt{y-1} = f^{-1}(x)$

$D_{f^{-1}} = V_f$

$y = x^2 + 4x + 5$

$y = (x+2)^2 + 1$

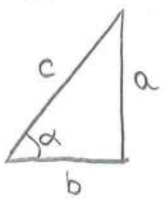


Ur bilden ser vi

$D_{f^{-1}} = V_f = [1, \infty[$

Föreläsning 2015-09-21

8,4. Trigonometriska funktionen för vinklar $0 < \alpha < 90°$



$\sin\alpha = \frac{a}{c}$

$\cos\alpha = \frac{b}{c}$

$\tan\alpha = \frac{a}{b}$

$\cot\alpha = \frac{b}{a}$

Trigonomiska ettan

$\boxed{\sin^2\alpha + \cos^2\alpha = 1}$

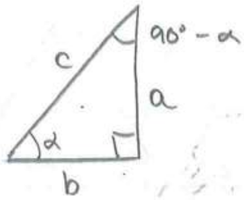
$\Rightarrow \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2 + b^2}{c^2}$

$= \frac{c^2}{c^2} = 1$

Pythagoras sats ger $a^2 + b^2 = c^2$

Ex. $\tan\alpha \cdot \cot\alpha = \frac{a}{b} \cdot \frac{b}{a} = 1$

Ex.



$\sin\alpha = \frac{a}{c}$

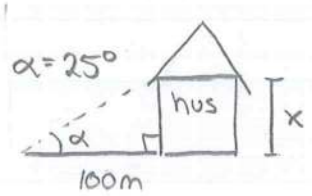
$\cos(90° - \alpha) = \frac{a}{c}$

$\sin\alpha = \cos(90° - \alpha)$

Analogt har vi:

$\cos\alpha = \sin(90° - \alpha)$

Ex. Bestäm höjden av ett hus

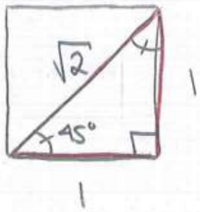


Lösning

$\frac{x}{100} = \tan 25°$

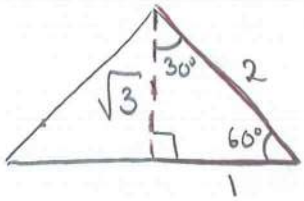
$\therefore\ x = 100 \cdot \tan 25° \approx 100 \cdot 0{,}47 \approx 47\text{m}$

Påstående:



Halv kvadrat

$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ } exempel på hur man har användning av sambandet.



Halv liksidig triangel

exempel $\cos 60° = \frac{1}{2}$, $\tan 30° = \frac{1}{\sqrt{3}}$

α	30°	45°	60°
sin α	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos α	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$

Där $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ o $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$

Samband mellan grader och radianer

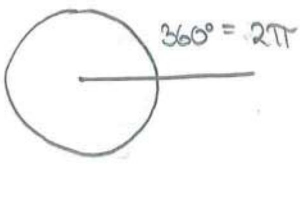
$\pi = 3{,}1415926\ldots$ radian

$180° = \pi$

$1° = \frac{\pi}{180°}$ $1 \text{ rad} = \frac{180°}{\pi}$

Ex. ① $90° = 90 \cdot 1° = 90 \cdot \frac{\pi}{180} = \frac{\pi}{2}$

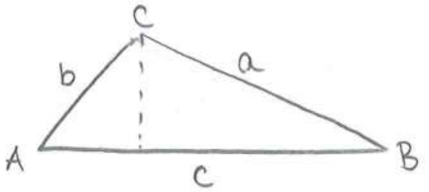
② $2\pi = 2 \cdot 180° = 360°$



③ $\frac{\pi}{2} = 90°$ $\frac{\pi}{3} = 60°$ $\frac{\pi}{4} = 45°$

$\frac{\pi}{6} = 30°$

T.3

För  gäller

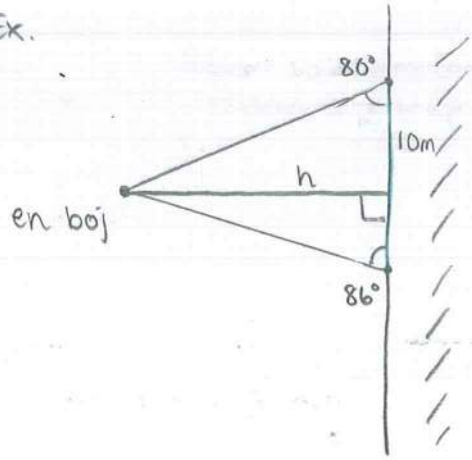
Sats 1-3 ① Area = $\frac{1}{2}bc \cdot \sin A$

② $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ (sinussatsen)

$\frac{1}{\cancel{2}} b\cancel{c} \cdot \sin A = \frac{1}{\cancel{2}} a\cancel{c} \cdot \sin B \Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b}$

③ $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

Ex.



Räkna ut avståndet h mellan kajen och bojen.

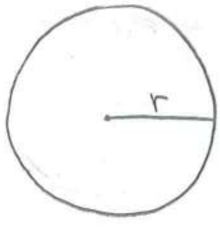
Lösning: Sinussatsen ger $\frac{x}{\sin 80°} = \frac{10}{\sin 14°}$

$x = \frac{10}{\sin 14°} \cdot \sin 80°$

Men $\frac{h}{x} = \sin 86°$

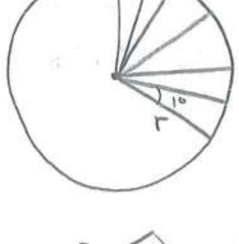
$h = x \cdot \sin 86° = \frac{10}{\sin 14°} \cdot \sin 80° \cdot \sin 86° \approx 40{,}6\,m$

T.4 ①



Arean = πr^2

②



360 stycken lika stora cirkelsektorer med arean:

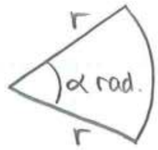
$\frac{\pi r^2}{360}$

③

cirkelsektor har arean

$\pi r^2 \cdot \frac{\beta}{360}$

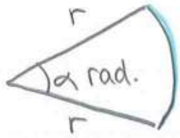
④



cirkelsektorn har arean

$$\pi r^2 \cdot \frac{\alpha}{2\pi} = \frac{1}{2} r^2 \cdot \alpha \leftarrow \text{radian}$$

⑤



Cirkelbågen har längden $\ell = 2r\pi \cdot \dfrac{\alpha}{2\pi} = r\alpha$

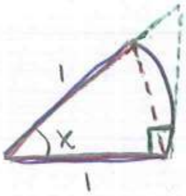
$\boxed{\text{dvs. } \ell = r \cdot \alpha} \leftarrow$ radian

Sats 5 sidan 66 i geometriboken

$$\boxed{\sin x < x < \tan x \quad \text{då } 0 < x < \frac{\pi}{2}}$$

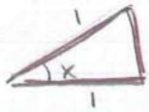
↑ radian

Bevis

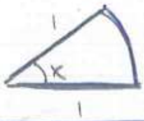


$\dfrac{h}{1} = \tan x$ dvs. $h = \tan x$

Klart att: Arean av



$<$ arean av



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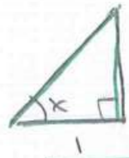


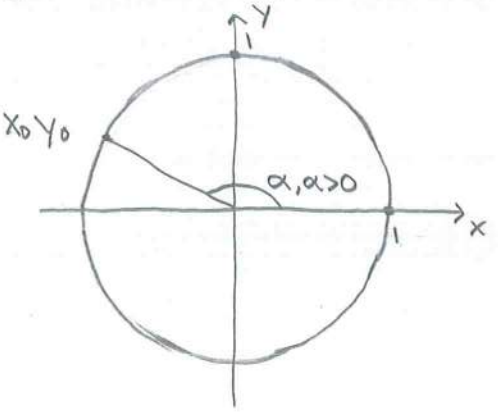
inre triangel | cirkelsektor | yttre triangel

dvs. $\frac{1}{2} \cdot 1 \cdot 1 \cdot \sin x < \frac{1}{2} \cdot 1 \cdot 1 \; x < \frac{1}{2} \, 1 \cdot \tan x \Rightarrow \underline{\sin x < x < \tan x}$

V.S.V.

Föreläsning 2015-09-23

Trigonometriska funktioner för vinklar $-\infty < \alpha < \infty$



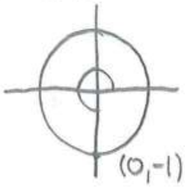
Def. $\sin \alpha = y_0$

$\cos \alpha = x_0$

$\tan \alpha = \dfrac{\sin \alpha}{\cos \alpha}$ då $\cos \alpha \neq 0$

$\cot \alpha = \dfrac{\cos \alpha}{\sin \alpha}$ då $\sin \alpha \neq 0$

Ex. $\sin 270^\circ = -1$



$(0,-1) = (\cos 270^\circ, \sin 270^\circ)$

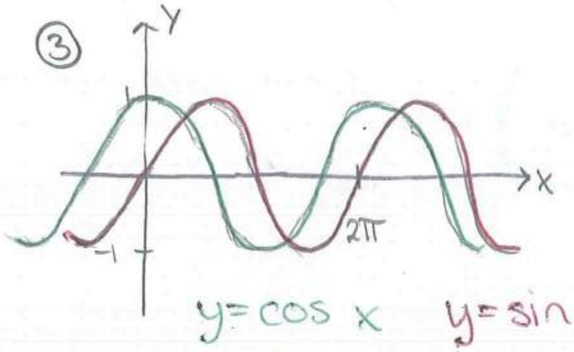
Egenskaper

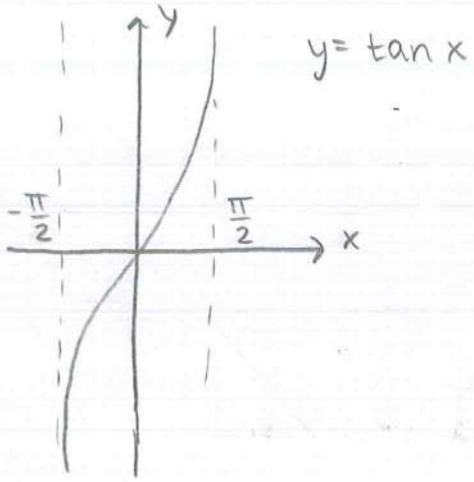
① $\sin x$ och $\cos x$ är periodiska funktioner med perioden 2π, dvs

$$\sin(x+2\pi) = \sin x$$
$$\cos(x+2\pi) = \cos x$$

② $$\tan(x+\pi) = \tan x$$
$$\cot(x+\pi) = \cot x$$

③





Trigonometriska formler

Sats ① $\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$

② $\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$

Följdsats ① $\cos^2 x + \sin^2 x = 1$

$\Rightarrow \cos x \cdot \cos x + \sin x \cdot \sin x = \cos(x-x)$
$= \cos 0 = 1$

② $\sin(\pi - x) = \underbrace{\sin \pi}_{0} \cdot \cos x - \underbrace{\cos \pi}_{-1} \cdot \sin x = \sin x$

③ $\sin(2x) = \sin(x+x) = \sin x \cdot \cos x + \cos x \cdot \sin x = 2 \sin x \cdot \cos x$

④ $\sin(-x) = \sin(0-x) = \underbrace{\sin 0}_{0} \cdot \cos x - \underbrace{\cos 0}_{1} \cdot \sin x = -\sin x$

⑤ $\cos(-x) = \ldots = \cos x$ (jämn funktion)

Problem Lös $\sin x = A$, där A är ett givet tal och $-1 \le A \le 1$

Sats Låt $\sin x_0 = A$, så är lösningar till $\sin x = A$

$x = x_0 + 2k\pi$ där $k = 0, \pm 1, \pm 2, \ldots$ och $x = \pi - x_0 + 2k\pi$

Bevis Ur $\sin x = A$. och $\sin x_0 = A$ vet vi $\sin x = \sin x_0$

som ger $x = x_0 + 2k\pi$ eller $x = \pi - x_0$

Sats Låt $\cos x_0 = B$ Så har ekvationen $\cos x = B$

Lösningar $x = \pm x_0 + 2k\pi$ där $k = 0, \pm 1, \pm 2, \ldots$

Ex Lös $\sin 3x = \frac{1}{2}$

Lösning vi vet att $\sin \frac{\pi}{6} = \frac{1}{2}$ $\quad 3x = \frac{\pi}{6} + 2k\pi$ eller $3x = \pi - \frac{\pi}{6} + 2k\pi$

ger lösningar

$$\boxed{x = \frac{\pi}{18} + \frac{2}{3}k\pi \quad \underline{o} \quad x = \frac{5\pi}{18} + \frac{2}{3}k\pi}$$

OBS! I beviset använder vi $\sin a = \sin b \iff a = b + 2k\pi$ ($360°k$)

$a = \pi - b + 2k\pi$

Ex. $\sin(2x) = \sin x$ Lösning $2x = x + 2k\pi$ $\quad x = 2k\pi$

$2x = \pi - x + 2k\pi$ $\quad x = \frac{\pi}{3} + \frac{2k\pi}{3}$

Sats ① Om $\tan x_0 = A$ där A $-\infty \leq A \leq \infty$ så har ekvationen

$\tan x = A$ lösningar $x = x_0 + k\pi$ där $k = 0, \pm 1, \pm 2, \ldots$

② Om $\cot x_0 = A$ så har ekvationen $\cot x = A$ lösningar

$x = x_0 + k\pi$ där $k = 0, \pm 1, \pm 2 \ldots$

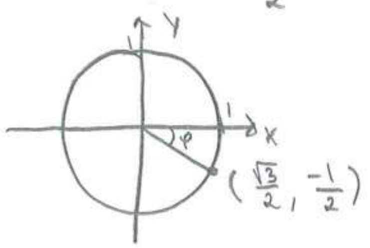
Ex. Lös $\sqrt{3}\sin x - \cos x = \sqrt{3}$

Lösning: $\sqrt{3}\sin x + (-1)\cos x = \sqrt{3} \iff \frac{\sqrt{3}}{(\sqrt{3})^2 + (-1)^2}\sin x + \frac{-1}{(\sqrt{3})^2 + (-1)^2}\cos x$

$= \frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2 + (-1)^2}}$ $\quad$ dvs $\quad \underbrace{\frac{\sqrt{3}}{2}}_{\cos\varphi}\sin x + \underbrace{\frac{(-1)}{2}}_{\sin\varphi}\cos x = \frac{\sqrt{3}}{2}$

sätt $\begin{cases} \cos\varphi = \frac{\sqrt{3}}{2} \\ \sin\varphi = \frac{-1}{2} \end{cases}$ där $\frac{\sqrt{3}}{2} + \left(\frac{-1}{2}\right)^2 = 1$ $\quad x^2 + y^2 = 1^2$

$\varphi = -\frac{\pi}{6}$ $\quad$ Så blir ekvationen:



$\cos\left(-\frac{\pi}{6}\right) \cdot \sin x + \sin\left(\frac{\pi}{6}\right) \cdot \cos x = \frac{\sqrt{3}}{2}$

$= \sin\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad \Longrightarrow$

$x - \frac{\pi}{6} = \frac{\pi}{3} + 2k\pi \qquad x - \frac{\pi}{6} = \pi - \frac{\pi}{3} + 2k\pi$

Lösningar:

$x = \frac{\pi}{2} + 2k\pi \qquad x = \frac{5\pi}{6} + 2k\pi$

Allmänt gäller: $a\sin x + b\cos x = c$

kan omskrivas $\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}}$

Arcussinus

Funktionen $f(x) = \sin x$, $-\infty < x < \infty$ saknar invers

men $g(x) = \sin x$ där $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ är injektiv

och har alltså inversen

$x = \arcsin y$, där $-1 \leq y \leq 1$

Ex. Bestäm inversen till $f(x) = 3 + \sin(2x)$, där $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

Lösning: $y = 3 + \sin(2x) \qquad y - 3 = \sin x$

Men $-\frac{\pi}{4} \leq 2x \leq \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\arcsin(y-3) = 2x \qquad x = \frac{1}{2}\arcsin(y-3)$

Den sökta inversen $f^{-1}(x) = \frac{1}{2}\arcsin(y-3) \qquad D_{f^{-1}} = V_f$

$D_{f^{-1}}: 2 \leq x \leq 4$

Def ① $y = \cos x$ där $0 \leq x \leq \pi$ har inversen $x = \arccos y$

$-1 \leq y \leq 1$

Def ② $y = \tan x$, där $-\frac{\pi}{2} < x < \frac{\pi}{2}$ har inversen

$x = \arctan y \qquad -\infty < y < \infty$

Def ③ $y = \cot x$ där $0 < x < \pi$ har inversen $x = \arctan y$

där $-\infty < x < \infty$