Exercise 6:

1) Let $Q(w) = \frac{1+\alpha w}{\beta + \gamma w}$, A is a matrix with known eigenvalues $\lambda_i[A]$.

$$Q(A) = (\beta I + \gamma A)^{-1}(I + \alpha A)$$

show that $\lambda[Q(A)] = Q(\lambda[A])$

$$\Rightarrow \quad (\beta \cdot I + \gamma A)^{-1}(I + \alpha A) \cdot \vec{v} = \lambda \vec{v}$$

$$(I \cdot \vec{v} + \alpha \cdot A\vec{v}) = \lambda(\beta \cdot I\vec{v} + \gamma A \cdot \vec{v}) \quad (1)$$

Let X be the matrix of eigenvectors to A. We can then write $\vec{v} = X \cdot \vec{v}'$, insert into (1):

$$(1 + \alpha \cdot \lambda[A])\cancel{X \cdot \vec{v}'} = \lambda[Q] \cdot (\beta + \gamma \cdot \lambda[A]) \cdot \cancel{X\vec{v}'}$$

$$\Rightarrow \frac{1 + \alpha \lambda[A]}{\beta + \gamma \cdot \lambda[A]} = \lambda[Q] \Leftrightarrow Q(\lambda[A]) = \lambda[Q(A)]$$

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2) let $B(\Delta t, \Delta x) = \left(I - \frac{\Delta t}{2} T_{\Delta x}\right)^{-1} \left(I + \frac{\Delta t}{2} T_{\Delta x}\right)$

The condition for stability would be $\|B\| \leq 1$

which we got if $|\lambda_k[B]| \leq 1$ for all k.

$$\lambda[T_{\Delta x}] = \frac{-4 \sin^2\left(\frac{k\pi \cdot \Delta x}{2}\right)}{\Delta x^2}$$

$$\Rightarrow |\lambda[B]| = \left| \frac{1 + \frac{\Delta t \lambda_k}{2}}{1 - \frac{\Delta t \lambda_k}{2}} \right| \leq 1$$

$$\left|1 + \frac{\Delta t}{2} \lambda_k\right| \leq \left|1 - \frac{\Delta t \lambda_k}{2}\right|$$

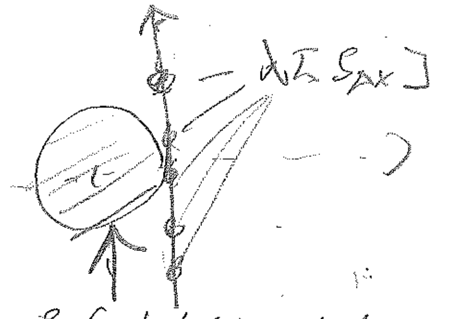
which is true for all $\lambda \in \mathbb{C}^-$ ($\mathbb{C}^-$ all complex numbers with $Re\{z\} \leq 0$)

This means no restriction on Δt.

3/ a) $u_{n+1} = (I + \Delta t \cdot S_{\Delta x}) u_n \Rightarrow |1 + \Delta t \cdot \lambda[S_{\Delta x}]| \leq 1$

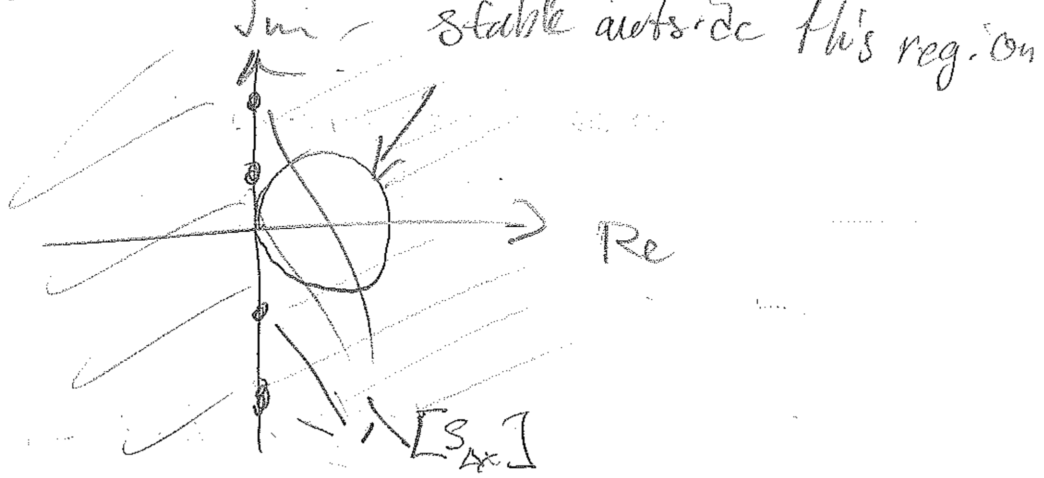
$2\sqrt{-1}\cos\left(\frac{k\pi}{n+1}\right) / 2\Delta x$

but $\lambda[S_{\Delta x}]$ has strictly imaginary λ, so this is always unstable.

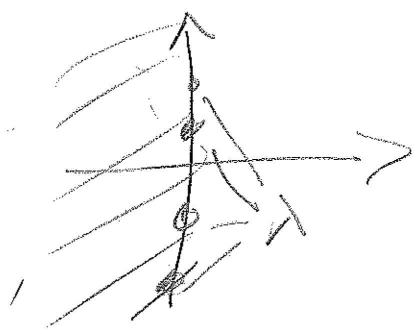


stability region for expl. euler

b) $\Rightarrow \dfrac{1}{|1 - \Delta t S_{\Delta x}|} \leq 1$, which is always true!



c) We already determined that this is stable for all $Re\{\lambda\} < 0$, so always stable!



stability region

4. Since this leads to the recurrance equation: $v_{n+1} = \lambda v_n$ will have decay if $|\lambda| < 1$, growth if $|\lambda| > 1$, so we need $|\lambda| = 1$.

Let $\lambda[S_{\Delta x}] = iw$, the TR gives:

$$|\lambda| = \left| \frac{1 + \frac{\Delta t}{2} iw}{1 - \frac{\Delta t}{2} iw} \right| = \frac{1 + \frac{\Delta t^2}{4} w^2}{1 + \frac{\Delta t^2}{4} w^2} = 1 \text{ ; so its unstable.}$$

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5) $U_{n+1} = U_{n-1} + 2\Delta t S_{\Delta x} U_n$. apply this to $y_t = \lambda y$.

a) Char-eq: $z^2 - 2\Delta t \lambda \cdot z - 1 = 0$

b) this can be written: $(z-\alpha)(z-\beta)$ (1)

where $\alpha\beta = -1$

c) a recurrance eq is stable if $|z_k| \leq 1$ if z_k is a root to the eq: $\alpha\beta = -1 \Leftrightarrow \alpha = \frac{-1}{\beta}$, which means that if one of the roots < 1

$\beta = -\frac{1}{\alpha}$ the other is larger than 1!

d) We can Write the roots as $\begin{cases} z_1 = e^{i\varphi} \\ z_2 = -e^{-i\varphi} \end{cases}$

from (1) we get: $z_1 + z_2 = 2\Delta t \lambda$

$$\Delta t \lambda = i \frac{e^{i\varphi} - e^{-i\varphi}}{i \cdot 2} = i \sin\varphi$$

But all roots $\neq 0$ must have multiplicity 1.

So $e^{i\varphi} \neq -e^{-i\varphi}$, $\varphi \neq \frac{\pi}{2}, \frac{3}{2}\pi \ldots$

and we get $\Delta t \lambda \in]-i, i[$

6) $\dot{u} = S_{\Delta x} u$, $S_{\Delta x} U = U \cdot \Lambda$ where U is the matrix of eigenvecs, Λ diagonal matrix with the vals.

let $\vec{u}_n = U \vec{v}_n$, $\vec{v}_n = U^{-1} u_n$

Insert into leapfrog: $u_{n+1} = u_{n-1} + 2\Delta t\, S_{\Delta x} u_n$

Where

$$U^{-1} u_{n+1} = U^{-1} u_n + 2\Delta t \cdot U^{-1} S_{\Delta x} U v_n$$

$$v_{n+1} = v_{n-1} + 2\Delta t\, \Lambda\, v_n$$

which gives us the eq.

$$w_{n+1} = w_{n-1} + 2\Delta t \lambda_k w_n$$

We know that $\lambda_k [S_{\Delta x}] = \frac{i \cos(k\pi \Delta x)}{\Delta x}$

and that $\Delta t \lambda$ need to be inside $]-i, i[$

$$\Rightarrow \left| \frac{\Delta t}{\Delta x} i \underbrace{\cos(k\pi / N+1)}_{\text{always } < 1} \right| < 1 \quad \text{for } k = 1 \dots N$$

$$\Rightarrow \frac{\Delta t}{\Delta x} \leq 1$$

b) yes, because we've run it on $u_t = u_x$ which is a conservation law, and the CFL condition is still fullfilled if $\Delta t_{new} = -\Delta t$ (backwards time)

7. This is simple to solve if you remember that

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{x} \longrightarrow e^{a} \quad \Rightarrow e^{\frac{i w_k \Delta t}{\Delta x}}$$

, if $\Delta x \to 0$, this will oscillate infinitely fast., it's not differentiable!

8. Let $u_t = -a u_x$

$$u(t+\Delta t) = u + \frac{\Delta t}{1!} u_t + \frac{\Delta t^2}{2!} u_{tt}$$

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x} \Leftrightarrow \frac{\partial}{\partial t} = -a \frac{\partial}{\partial x} \Leftrightarrow \frac{\partial^2}{\partial t^2} = a^2 \frac{\partial^2}{\partial x}$$

$$\Rightarrow u_{tt} = a^2 u_{xx}$$

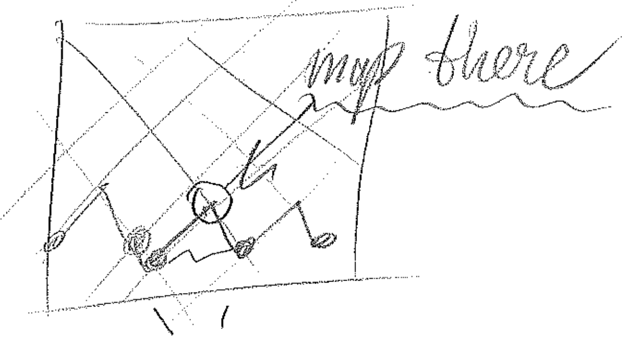
$$\Rightarrow u(t+\Delta t, x) = u_n - a \Delta t \, S_{\Delta x} u_n + a^2 \frac{\Delta t^2}{2} T_{\Delta x} u_n$$

9. Lax-Friedrichs:

$$u_j^{n+1} = \frac{(u_{j+1}^n + u_{j-1}^n)}{2} - a\Delta t \frac{(u_{j+1}^n - u_{j-1}^n)}{2\Delta x}$$

$$= \overbrace{\frac{1}{2}\left(1 + \frac{a\Delta t}{\Delta x}\right)}^{a} u_{j-1}^n + \overbrace{\frac{1}{2}\left(1 - \frac{a\Delta t}{\Delta x}\right)}^{b} u_{j+1} =$$

$$\Rightarrow T_{\Delta x} = \begin{bmatrix} 0 & b & 0 & \cdots & a \\ a & 0 & b & 0 \cdots & 0 \\ 0 & a & 0 & b \cdots & 0 \\ \vdots & & \ddots & \ddots & b \\ b & 0 & \cdots & a & 0 \end{bmatrix}$$



if $\frac{a\Delta t}{\Delta x} = 1$ we get:

$$T_{\Delta x} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

which just moves the solution.

$$\lambda_k[A(1)] = e^{2\pi i k/N}, \quad k = 1:N$$

which is stable! cause $|\lambda_k[A(1)]| \leq 1$