Numerical Methods for Differential Equations FMNN10/NUMN12
Repeat exam 2013-04-06 Results to be announced 2013-04-13
Solution sketch by Gustaf Söderlind

Exam duration 08:00 – 13:00. A minimum of 15 points out of 30 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(4p)** The special second order initial value problem $\ddot{y} = f(y)$ with initial conditions $y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$ models many problems in mechanics, e.g. planetary or satellite orbits. We are going to construct a method of the form

$$y_n - 2y_{n+1} + y_{n+2} = h^2\left(\beta_0 f(y_n) + \beta_1 f(y_{n+1}) + \beta_2 f(y_{n+2})\right).$$

   (a) Determine the coefficients β_i so that the order of consistency is maximal. What is the maximal order? (3p)

   (b) Is the method explicit or implicit? (1p)

2. **(6p)** Consider an implicit Runge-Kutta method with Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/3 & 2/3 \\ \hline b^{\mathrm{T}} & 1/3 & 2/3 \end{array}$$

   (a) Write the formulas associated with this method. (1p)

   (b) Find the stability function $R(h\lambda)$. (3p)

   (c) Is the method A-stable? (2p)

3. **(6p)** Consider the following nonlinear two-point boundary value problem:

$$y'' - 2yy' + y = g(x)$$
$$y(0) = \alpha, \qquad y'(1) = \beta.$$

(a) Introduce a suitable grid and discretize with a standard second order method. Give all details about the grid (number of grid points and their location, as well as mesh width Δx) and formulate the discretization. Include the boundary conditions in the equation system. (4p)

(b) Construct the Jacobian matrix associated with the system. (2p)

4. **(4p)** In the course we have studied numerical methods for Sturm–Liouville eigenvalue problems. One special type of such problems has the form

$$y'' + d(x)y = \lambda y$$

with boundary conditions $y(0) = y(1) = 0$, and the function $d(x) > 0$ on $[0, 1]$. Formulate this as an algebraic eigenvalue problem

$$Ay = \lambda_{\Delta x} y.$$

by using a *second order discretization*. Take care to treat the function $d(x)$ properly, and give the matrix A.

5. **(5p)** Consider the linear convection–diffusion equation

$$u_t = u_{xx} + u_x.$$

with homogeneous boundary conditions, and initial condition $u(0, x) = g(x)$.

(a) Introduce a suitable notation and write down a standard 2nd order method-of-lines discretization in space combined with the *trapezoidal rule* for time-stepping ("Crank–Nicolson's method"). (3p)

(b) As the method is implicit, one will have to solve a linear system of equations on each step. Write down this system in matrix–vector form. (2p)

6. **(5p)** The Lax–Friedrichs scheme for the advection problem with $a > 0$,

$$u_t + au_x = 0, \quad 0 \le x \le 1, \quad t \ge 0,$$
$$u(x, 0) = \phi_0(x),$$

is

$$u_j^{n+1} = (u_{j+1}^n + u_{j-1}^n)/2 - a\Delta t(u_{j+1}^n - u_{j-1}^n)/(2\Delta x).$$

Consider this problem for periodic boundary conditions, $u(t, 0) = u(t, 1)$.

(a) Let $\mu = \Delta t/\Delta x$ and write the resulting recursion in matrix–vector form,

$$U^{n+1} = T_\mu U^n + V^n$$

with $U^n = [u_1^n, u_2^n, \dots, u_N^n]^{\mathrm{T}}$. What are the matrix T_μ, the initial condition U^0 and the vector V^n? take care to define Δx in terms of N, the number of equations in the recursion. (3p)

(b) What is the CFL condition for this method? (No derivation is required.) (1p)

(c) Draw the computational stencil ("beräkningsmolekyl") for the method. (1p)

LYCKA TILL!
G.S.

Numerical Methods for Differential Equations FMNN10/NUMN12
Final exam 2012-12-19 Results to be announced 2012-12-22

Exam duration 08:00 – 13:00. A minimum of 15 points out of 30 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(5p)** The two-step "difference corrected BDF method" for the initial value problem $y' = f(y)$; $y(0) = y_0$ can be written

$$\left(\nabla + \frac{\nabla^2}{2}\right) y_n = \left(1 - \gamma\nabla^2\right) hf(y_n),$$

where ∇ is the backward difference operator, and γ is a constant (to be determined below).

   (a) Rewrite the formula as $\alpha_2 y_n + \alpha_1 y_{n-1} + \alpha_0 y_{n-2} = h\beta_2 f(y_n) + h\beta_1 f(y_{n-1}) + h\beta_0 f(y_{n-2})$. Determine the coefficients α_j and β_j. Express the latter in terms of γ. (2p)

   (b) Show that the method is zero-stable (convergent) for every γ. (1p)

   (c) For $\gamma = 0$ the method reduces to the standard two-step BDF method of order $p = 2$. Determine a nonzero value for γ such that the method has order of consistency $p = 3$. (Hint: note that $\gamma\nabla^2 P'(t) = 0$ if P' is a polynomial of degree at most 1; this means that the order is $p \geq 2$ for every γ, and that it is sufficient to consider polynomials P of degree 3 to determine γ.) (2p)

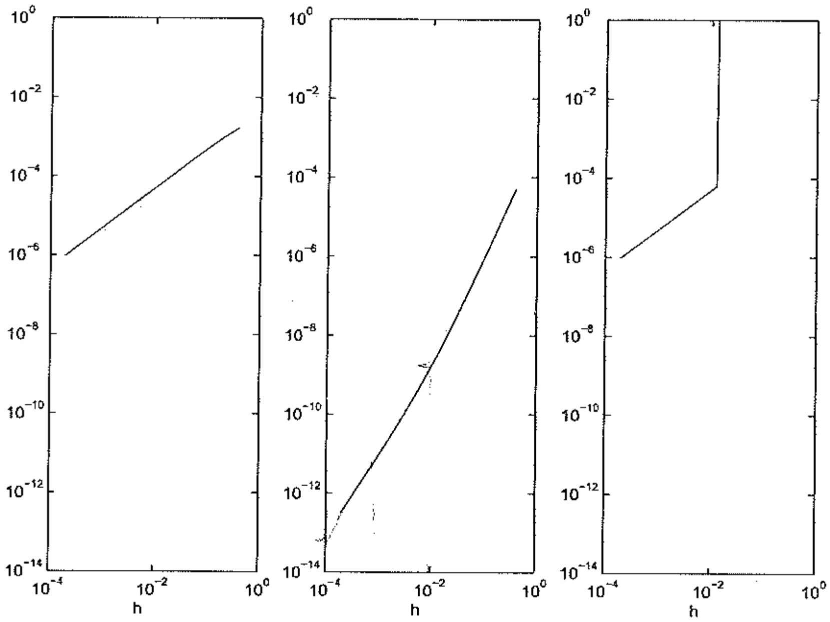
2. **(5p)** The θ method reads

$$y_{n+1} = y_n + h(\theta f(y_n) + (1-\theta) f(y_{n+1})).$$

The method was tested numerically, with three different values of θ, solving a moderately stiff problem with constant stepsize h.

   (a) Give the names and orders of the methods for $\theta = 0$, $1/2$, 1 and draw their respective stability regions. (2p)

   (b) In the plots below, identify which method (what value of θ) corresponds to each plot, and motivate your answer carefully giving some theoretical argument in each case. The plots show the absolute error at the endpoint vs. the stepsize h. (3p)



3. **(6p)** Construct a 2nd order discretization for the nonlinear two-point boundary value problem

$$y'' + (y^3)' + y = \sin \pi x$$
$$y(0) = 1, \qquad y'(1) = 2.$$

Introduce a grid and give all details of how you have constructed it by making a clear drawing, including defining the number of equations as well as Δx. Write down the resulting equation system, including the boundary conditions. Simplify as far as possible, keeping only the unknowns in the left-hand side.

4. **(4p)** In Euler buckling of a beam of length L under an axial load P, the deflection u of the beam's center line is given by the equation

$$u'' = \frac{P}{EI} u$$

with various boundary conditions depending on which buckling case is considered, and where EI is the product of Young's modulus of elasticity E and the beam's cross-sectional moment of inertia I.

Here we shall consider the first two cases

$$(1) \qquad u'(0) = 0, \qquad u(L) = 0$$

and

$$(2) \qquad u(0) = 0, \qquad u(L) = 0.$$

Construct the linear algebraic eigenvalue problem that results from a second order discretization of this problem. In particular, give the matrix for each buckling case, and show clearly what the difference is between the matrices in the two cases, i.e., how are the matrix elements affected by changing the boundary conditions, and how are the matrix dimensions or other defining properties of the eigenvalue problem affected?

5. **(5p)** Consider the following PDEs for $t \geq 0$ and $x \in [0, 1]$:

   (a) $u_t = d \cdot u_{xx} + f(u)$

   (b) $u_t + \frac{1}{2}(u^2)_x = 0$

   (c) $u_t + a \cdot u_x = d \cdot u_{xx}$

   (d) $u_{tt} = c^2 \cdot u_{xx}$

   (e) $u_t + uu_x = u_{xx}$

   For each of the cases above, give the name of the equation and classify it as elliptic, parabolic or hyperbolic.

6. **(5p)** Consider the Klein–Gordon equation

$$u_{tt} = u_{xx} - u,$$

   with periodic boundary conditions $u(t, 0) = u(t, 1)$ and $u_x(t, 0) = u_x(t, 1)$, and initial condition $u(0, x) = \sin^2 2\pi x$.

   (a) Introduce a suitable grid (make a drawing) and clarify at what points you solve for the unknowns, as well as how the number of unknowns, N, is related to Δx. Write down a standard symmetric 2nd order discretization in space, and state the resulting method-of-lines ordinary differential equation in vector-matrix form. (3p)

   (b) Combine with an explicit symmetric discretization of u_{tt} in time, to get an explicit two-step, 2nd order method for time-stepping, and write down the full discretization. Also give the structure of the CFL condition in the form $\Delta t / \Delta x^p \leq C$, where you only have to give the value of p. (2p)

LYCKA TILL — GOOD LUCK! G.S.

GOD JUL – MERRY CHRISTMAS – JOYEUX NOËL – FELIZ NAVIDAD – FRÖHLICHE WEIHNACHTEN – BUON NATALE – BON NADAL

Exkurla 18/12-2011

1. $y' = f(y) \quad y(0) = y_0$

$$\left(\nabla + \frac{\nabla^2}{2}\right) y_n = (1 - \gamma \nabla^2)\, h f(y_n)$$

$\nabla y_n = y_n - y_{n-1} \qquad \nabla^2 = y_n - 2y_{n-1} + y_{n-2}$

$$y_n - y_{n-1} + \frac{1}{2} y_n - y_{n-1} + \frac{1}{2} y_{n-2} = hf(y_n) - \gamma h f(y_n) + 2\gamma h f(y_{n-1}) - \gamma h f(y_{n-2})$$

$\Rightarrow \alpha_0 = \frac{1}{2} \quad \alpha_1 = -2 \quad \alpha_2 = \frac{3}{2}$

$\beta_0 = -\gamma \quad \beta_1 = 2\gamma \quad \beta_2 = 1 - \gamma$

$$\frac{3}{2} y_n - 2y_{n-1} + \frac{1}{2} y_{n-2} = h(1-\gamma) f(y_n) + 2\gamma h f(y_{n-1}) - \gamma h f(y_{n-2})$$

b) For zerostability the method must fulfill the root condition.

$\Rightarrow$ Characteristic polynomial: $\frac{3}{2} z^2 - 2z + \frac{1}{2} = 0 \Rightarrow$ {independent of γ so applies for all γ!}.

$\Rightarrow z^2 - \frac{4}{3} z + \frac{1}{3} = 0 \Rightarrow \left(z - \frac{2}{3}\right)^2 + \frac{1}{3} - \frac{4}{9} = 0 \Rightarrow z - \frac{2}{3} = \pm \frac{1}{3}$

$\Rightarrow z = \frac{2}{3} \pm \frac{1}{3} \Rightarrow \begin{cases} z_1 = 1 \\ z_2 = \frac{1}{3} \end{cases}$ We have simple moduli and both modulus $|z_i| \leq 1$.

c) We set $y = P(t) = t^m$ and $y' = f(y) = P'(t) = mt^{m-1}$ and also $t_n = 2h \quad t_{n-1} = h \quad t_{n-2} = 0$.

$m = 3$

$$\frac{3}{2} \cdot (2h)^3 - 2 \cdot h^3 + 0 = h(1-\gamma) \cdot 3 \cdot (2h)^2 + 2\gamma \cdot h \cdot 3 \cdot h^2$$

$$\Rightarrow 12h^3 - 2h^3 = (h - h\gamma) 12h^2 + 6\gamma h^3$$

$$\Rightarrow 10h^3 = 12h^3 - 12\gamma h^3 + 6\gamma h^3$$

$$10h^3 = h^3(12 - 6\gamma) \Rightarrow 12 - 6\gamma = 10$$

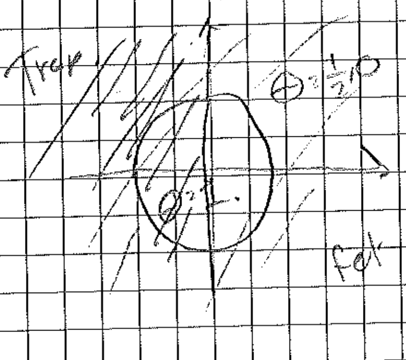
$$\Rightarrow 6\gamma = 2 \Rightarrow \gamma = \frac{1}{3}$$ 5p.

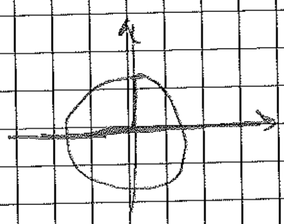
2. a) $\Theta = 0$ Implicit euler method

$\Theta = \frac{1}{2}$ Trapezoidal rule

$\Theta = 1$ Explicit euler method

rätt OK?





Trap: left halfplane.

Expl: $|z+1| \leq 1$

Impl: $|z-1| \geq 1$

b) The middle is the only 2nd order method $\Rightarrow \theta = \frac{1}{2}$.
The right one shows instability at $h = 10^{-2}$ $\Rightarrow$ explicit since it has stability bounds on h $\Rightarrow \theta = 1$.

The left one is an implicit first order method $\Rightarrow \theta = 0$.

3. $y'' + (y^3)' + y = \sin \pi x \qquad y(0) = 1 \qquad y'(1) = 2.$

We have a Neumann condition at the right endpoint so we introduce our steps as $\Delta x = \frac{1}{N + \frac{1}{2}}$ with N interior grid points, and that $x_j = j \cdot \Delta x$. We approximate the Neumann condition $y'(1) = 2$ by

$$\frac{y_{n+1} - y_n}{\Delta x} = 2 \Rightarrow y_{n+1} - y_n = 2\Delta x \Rightarrow$$
$$\Rightarrow y_{n+1} = 2\Delta x + y_n$$

The discretization reads:

$$\frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} - \frac{y_{n-1}^3 - y_{n+1}^3}{2\Delta x} + y_n = \sin(\pi x_n)$$

So

$$\frac{1 - 2y_1 + y_2}{\Delta x^2} - \frac{1 - y_2^3}{2\Delta x} + y_1 = \sin(\pi x_1)$$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} - \frac{y_{i-1}^3 - y_{i+1}^3}{2\Delta x} + y_i = \sin(\pi x_i) \qquad i = 2 : N-1$$

$$\frac{y_{n-1} - 2y_n + 2\Delta x + y_n}{\Delta x^2} - \frac{y_{n-1}^3 - (2\Delta x + y_n)^3}{2\Delta x} + y_n = \sin(\pi x_n)$$

And after simplifying: $(x_j = j\Delta x)$

$$\frac{y_2 - 2y_1}{\Delta x^2} + \frac{y_2^3}{2\Delta x} + y_1 = \sin(\pi\Delta x) - \frac{1}{\Delta x^2} + \frac{1}{2\Delta x}$$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} - \frac{y_{i-1}^3 - y_{i+1}^3}{2\Delta x} + y_i = \sin(\pi j \Delta x) \qquad j = 2 : N-1$$

$$\frac{y_{n-1} - y_n}{\Delta x^2} - \frac{y_{n-1}^3 - 8\Delta x^3 y_n + 4\Delta x y_n^2 - y_n^3}{2\Delta x} + y_n = \sin(\pi N \Delta x) - \frac{2}{\Delta x} + 4\Delta x^2$$

6p

4. $u'' = \frac{P}{EI} u$. (1) $u'(0)=0$, $u(1)=0$ (2) $u(0)=0$, $u(1)=0$

We have $\frac{u_{n-1}-2u_n+u_{n+1}}{\Delta x^2} = \frac{P}{EI} u_n$ We will have $A_{\Delta x} U = u$.

The first case we have a Neumann condition at the left endpoint, as in the previous exercise we will then have $\Delta x = \frac{1}{N+\frac{1}{2}}$, $x_i = i \cdot \Delta x$

and the condition $\frac{u_1 - u_0}{\Delta x} = 0 \Rightarrow u_1 = u_0$.

So we have (1) $u_1 = u_0$, $u_{N+1} = 0$

$$\frac{-u_1 + u_2}{\Delta x^2} = \frac{P}{EI} u_1$$

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} = \frac{P}{EI} u_j \quad j = 2:N-1. \quad A^{(1)}_{\Delta x} = \frac{1}{\Delta x^2} \underset{(N \times N)}{\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ & \ddots & \\ & 1 & -2 \end{pmatrix}}$$

$$\frac{u_{N-1} - 2u_N}{\Delta x^2} = \frac{P}{EI} u_N$$

And (2) $u_0 = 0$, $u_{N+1} = 0$. And also $\Delta x = \frac{1}{N}$ and $x_i = i \cdot \Delta x$

$$\frac{-2u_1 + u_2}{\Delta x^2} = \frac{P}{EI} u_1$$

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} = \frac{P}{EI} u_i \quad A^{(2)} = \frac{1}{\Delta x^2} \underset{(N \times N)}{\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ & \ddots & \\ & 1 & -2 \end{pmatrix}}$$

$$\frac{u_{N-1} - 2u_N}{\Delta x^2} = \frac{P}{EI} u_N$$

So, both problems are similar, but the Neumann condition requires the first matrix to be modified on the first line. Also, the grid is changed.

4p.

5 a) reaction-diffusion ~~hyperbolic~~ parabolic.

b) $u_t = \frac{1}{2}(u^2)_x = u u_x$ Inviscid Burger hyperbolic. 1p

c) $u_t + a \cdot u_x = d \cdot u_{xx}$ Convection Diffusion parabolic

d) $u_{tt} = c^2 u_{xx}$ Wave hyperbolic 1p.

e) $u_t + u u_x = u_{xx}$ Viscous Burger ~~hyperbolic~~ parabolic

6. $u_{tt} = u_{xx} - u.$

$u(t,0) = u(t,1)$

$u_x(t,0) = u_x(t,1)$

$u(0,x) = \sin^2(2\pi x).$

Kollar på fre.

EXTENTA #1.

Numerical Methods for Differential Equations FMNN10, 091218
Solution sketch by Gustaf Söderlind

Exam duration 08:00 – 13:00. A minimum of 15 points out of 30 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, textbooks, lecture notes or any other electronic or written material may be used during the exam.

1. **(5p)** For the initial value problem $\dot{y} = f(y)$ the *explicit midpoint method* reads
$$y_{n+1} - y_{n-1} = 2\,\Delta t \cdot f(y_n).$$
   (a) Find the method's order of consistency. (2p)
   (b) Show that the method is zero-stable. (1p)
   (c) Apply the method to the linear test equation $\dot{y} = \lambda y$ and show that both roots of the characteristic equation must have exactly unit modulus (absolute value 1) for the method to be stable. (2p)

Solution. For the order of consistency, we try whether the formula holds for $y(t) = t^p$ and $f(y) = pt^{p-1}$ and take $t_{n-1} = -\Delta t$, $t_n = 0$, and $t_{n+1} = \Delta t$. For $p = 0 : 3$ we get

$$\begin{aligned} 1 - 1 &= 2\,\Delta t \cdot 0 \\ \Delta t - (-\Delta t) &= 2\,\Delta t \cdot 1 \\ \Delta t^2 - (-\Delta t)^2 &= 2\,\Delta t \cdot 2 \cdot 0^1 \\ \Delta t^3 - (-\Delta t)^3 &\neq 2\,\Delta t \cdot 3 \cdot 0^2. \end{aligned}$$

The formula holds for $p = 0, 1$ and 2, but breaks down for $p = 3$ so the order of consistency is $p = 2$.

For zero-stability, let $f(y) \equiv 0$. The difference equation $y_{n+1} - y_{n-1} = 0$ has characteristic equation $z^2 - 1 = 0$ with roots $z = \pm 1$. They are simple, and since $|z| \leq 1$ the method is zero stable.

The linear test equation gives $y_{n+1} - 2\Delta t\lambda y_n - y_{n-1} = 0$ with characteristic equation $z^2 - 2\Delta t\lambda z - 1 = 0$. The product of the two roots is $z_1 z_2 = -1$. Thus, if one root is smaller than one in magnitude, the other is greater than one. The only possibility is that both have modulus 1.

2. **(5p)** Consider the 3-stage Runge-Kutta method with Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 1/4 & 3/4 & 0 \\ \hline & \alpha & 1-2\alpha & \alpha \end{array}.$$

(a) Find its stability function $R(h\lambda)$. (3p)

(b) For what value of α is the method of order 3? (1p)

(c) Is the method A-stable? (1p)

Solution. For the linear test equation we get

$$\begin{aligned} h\dot{Y}_1 &= h\lambda \cdot y_n \\ h\dot{Y}_2 &= h\lambda \cdot (y_n + h\dot{Y}_1/2) = h\lambda \cdot (1 + h\lambda/2) \cdot y_n \\ h\dot{Y}_3 &= h\lambda \cdot (y_n + h\dot{Y}_1/4 + 3h\dot{Y}_2/4) = \\ &= h\lambda \cdot (1 + h\lambda/4 + 3h\lambda/4 + 3(h\lambda)^2/8) \cdot y_n \\ y_{n+1} &= y_n + \alpha(h\dot{Y}_1 + h\dot{Y}_3) + (1-2\alpha)h\dot{Y}_2 = \\ &= (1 + h\lambda + (h\lambda)^2/2 + 3\alpha(h\lambda)^3/8) \cdot y_n \\ R(h\lambda) &= 1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{3\alpha(h\lambda)^3}{8}, \end{aligned}$$

so the method is of order $p = 3$ for $\alpha = 4/9$, which makes the last term equal to $(h\lambda)^3/6$. Since the method is explicit, the stability function is a polynomial; therefore the method cannot be A-stable.

3. **(5p)** Three students, the Good, the Bad and the Ugly, try to solve an eigenvalue problem $u'' = \lambda u$ with a Dirichlet condition $u(0) = 0$ and a Neumann condition $u'(1) = 0$, and are trying hard to get second order accuracy as $\Delta x \to 0$. They try their programs, obtaining different approximations to the eigenvalue near $-\pi^2/4$.

The **Good Student** uses an $N \times N$ matrix

$$T_{\Delta x} = \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & \dots & 0 \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -1 \end{pmatrix}$$

with $\Delta x = 1/(N + 1/2)$.

The **Bad Student**, on the other hand, claims that convergence "has nothing to do with how you define Δx as it varies and tends to zero anyway," and uses the same $N \times N$ matrix as the Good Student, except for putting $\Delta x = 1/(N + 1)$ "as usual" in his program.

While the Good and the Bad are disputing the importance of Δx, the **Ugly Student** unexpectedly enters, telling the Bad Student:

"There are two kinds of students, my friend. Those who took the course FMNN10, and those who talk. You talk."

But if the Bad really insists on using $\Delta x = 1/(N+1)$, the Ugly can fix the program for him to achieve second order. The Ugly Student sets out to do this and changes the program in *two places*. First, he redefines the matrix to be $(N+1) \times (N+1)$, while *keeping* $\Delta x = 1/(N+1)$, and second, he changes the last row of the matrix so that it reads

$$T_{\Delta x} = \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & \dots & 0 \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 2 & -2 \end{pmatrix}.$$

The graphs of the error when the three students' codes were run are shown in Figure 3, possibly but not necessarily in order.

(a) Give a simple argument to indicate which student may have obtained what graph. (2p)

(b) Explain how the Ugly Student obtains 2nd order. (3p)

Solution. The Good Student uses standard procedure to represent the Neumann boundary condition to 2nd order. Using the same matrix as the Good Student, the Bad Student has the wrong step size, and produced the middle graph, which is only 1st order. The other two show 2nd order convergence.

One cannot tell whether the Good produced the top graph and the Ugly the one at the bottom, or the other way around. Either way, the Ugly Student got away with 2nd order, sneering at the Bad Student:

"If you need 2nd order, make it 2nd order. Don't talk."

How did the Ugly Student do it? He uses a standard grid with N internal points and $\Delta x = 1/(N+1)$, so *he also has to solve for* u_{N+1} at $x_{N+1} = 1$. That means he needs an $(N+1) \times (N+1)$ matrix, and explains why he changed the dimension of the matrix.

He then represented the Neumann condition to 2nd order at $x_{N+1} = 1$, by using the approximation

$$u'(1) \approx \frac{u_{N+2} - u_N}{2\Delta x} = 0$$

implying that $u_{N+2} = u_N$. Upon inserting this into his last equation,

$$\frac{u_N - 2u_{N+1} + u_{N+2}}{\Delta x^2} = \lambda u_{N+1}$$

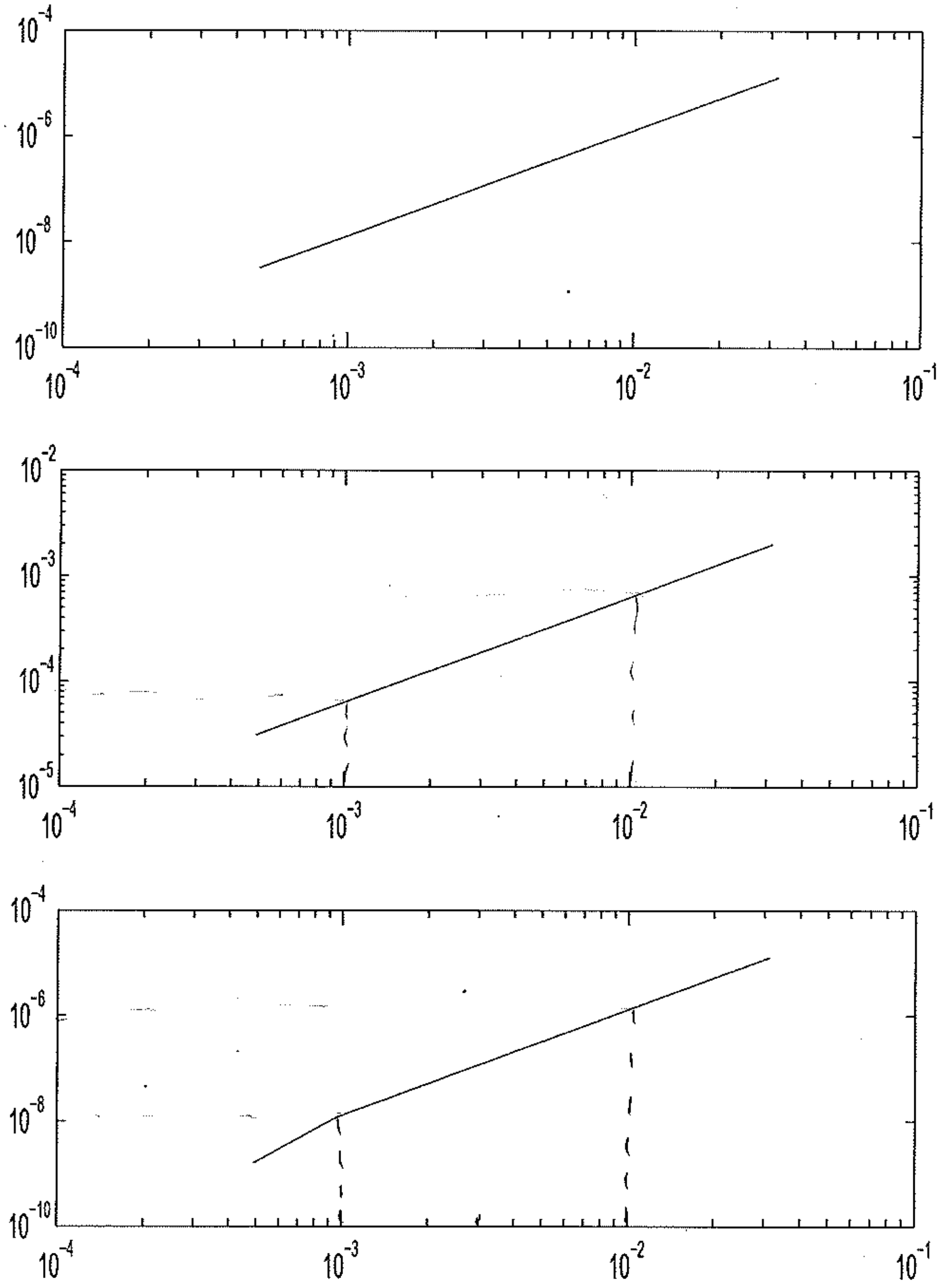


Figure 1: Error, as a function of Δx, in the determination of the eigenvalue near $-\pi^2/4$ for three methods, devised by three different students. Note that vertical scales are different in the graphs.

he gets the last row

$$\frac{2u_N - 2u_{N+1}}{\Delta x^2} = \lambda u_{N+1}.$$

This completes his method.

4. **(5p)** In order to determine the shape of a circular drum skin's oscillations, one needs to solve the *Bessel equation*

$$-u'' - \frac{u'}{x} = \lambda u$$

with boundary conditions $u'(0) = 0$, $u(R) = 0$, where R is the radius of the drum.

Construct a second order discretization of this problem to formulate it as a linear algebraic eigenvalue problem. Include all important details on grid point locations, mesh size Δx, boundary conditions etc., so that the problem is ready for programming. For full credit, the method must be 2nd order accurate.

Solution. A grid with N internal points and $\Delta x = R/(N + 1/2)$ with grid points located at $x_k = \Delta x/2 + (k-1)\Delta x$ for $k = 1 : N$ is constructed. Note that *one cannot have a grid point at $x = 0$ because the second term of the differential equation is divided by x.* Further, note that the Neumann condition at $x = 0$ only makes that natural.

We discretize with symmetric finite differences:

$$\begin{aligned}
-\frac{u_0 - 2u_1 + u_2}{\Delta x^2} - \frac{-u_0 + u_2}{2x_1 \Delta x} &= \lambda_{\Delta x} u_1 \\
-\frac{u_{k-1} - 2u_k + u_{k+1}}{\Delta x^2} - \frac{-u_{k-1} + u_{k+1}}{2x_k \Delta x} &= \lambda_{\Delta x} u_k \\
-\frac{u_{N-1} - 2u_N + u_{N+1}}{\Delta x^2} - \frac{-u_{N-1} + u_{N+1}}{2x_N \Delta x} &= \lambda_{\Delta x} u_N.
\end{aligned}$$

Next, we insert the boundary conditions, $u_{N+1} = 0$ and $u_0 = u_1$, the latter being a 2nd order approximation to the Neumann condition at the origin. This gives the final system,

$$\begin{aligned}
-\frac{-2u_1 + 2u_2}{\Delta x^2} &= \lambda_{\Delta x} u_1 \\
-\frac{u_{k-1} - 2u_k + u_{k+1}}{\Delta x^2} - \frac{-u_{k-1} + u_{k+1}}{2x_k \Delta x} &= \lambda_{\Delta x} u_k \\
-\frac{u_{N-1} - 2u_N}{\Delta x^2} - \frac{-u_{N-1}}{2x_N \Delta x} &= \lambda_{\Delta x} u_N,
\end{aligned}$$

where, in the first equation, we have used the fact that $x_1 = \Delta x/2$, and $u_0 = u_1$ before collecting terms. Noting that $2x_k = (2k-1)\Delta x$, we can also write it in tridiagonal matrix form, $A_{\Delta x}u = \lambda_{\Delta x}u$, where $A_{\Delta x}$ is the *drumskin matrix*

$$A_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} 2 & -2/1 & & \dots & 0 \\ -2/3 & 2 & -4/3 & & \\ & -4/5 & 2 & -6/5 & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & & -\frac{2N-2}{2N-1} & 2 \end{pmatrix}.$$

5. **(5p)** For $t \geq 0$ and $x \in [0, 1]$, write down the following PDEs:

   (a) Reaction-diffusion equation

   (b) Advection equation

   (c) Convection-diffusion equation

   (d) Viscous Burgers equation

   (e) Wave equation

   For each equation, state whether the problem is *elliptic, parabolic or hyperbolic*, and, assuming that an explicit time stepping method is used, give the *CFL condition* in the form $\Delta t/\Delta x^p \leq C$ that will be necessary for stability and convergence. (You only have to give the power p, not the value of the constant C.)

   Solution. The (sample) equations are

   (a) $u_t = u_{xx} + f(u)$; parabolic; $p = 2$

   (b) $u_t + u_x = 0$; hyperbolic; $p = 1$

   (c) $u_t + u_x = u_{xx}$; parabolic; $p = 2$

   (d) $u_t + uu_x = u_{xx}$; parabolic; $p = 2$

   (e) $u_{tt} = u_{xx}$; hyperbolic; $p = 1$

6. **(5p)** The Lax–Friedrichs scheme for the advection problem with $a > 0$ and periodic boundary conditions,

$$\begin{aligned} &u_t + au_x = 0, \quad 0 \leq x \leq 1, \quad t \geq 0, \\ &u(x, 0) = \phi_0(x), \end{aligned}$$

   is

$$u_j^{n+1} = (u_{j+1}^n + u_{j-1}^n)/2 - a\Delta t(u_{j+1}^n - u_{j-1}^n)/(2\Delta x).$$

   (a) Let $\mu = \Delta t/\Delta x$. Rewrite this scheme as a matrix–vector recursion $U^{n+1} = T_\mu U^n + V^n$ with $U^n = [u_1^n, u_2^n, \dots, u_L^n]^{\mathrm{T}}$, giving the matrix T_μ and the vector V^n. Is the matrix symmetric, skew-symmetric, unsymmetric, circulant, or of some other type? (3p)

(b) Draw the computational stencil ("beräkningsmolekyl") for the Lax–Friedrichs method with $\mu = 1/a$, inserting the correct values that correspond to the particular choice of μ above. Is the method an upwind or a downwind method for this value of μ? (2p)

Solution. Let $a_+ = (1 + a\mu)/2$ and $a_- = (1 - a\mu)/2$. Then

$$U^{n+1} = \begin{pmatrix} 0 & a_- & \dots & a_+ \\ a_+ & 0 & a_- & \vdots \\ & & \ddots & \\ a_- & \dots & a_+ & 0 \end{pmatrix} U^n = T_\mu U^n,$$

where the element a_- goes on the superdiagonal and in the lower left corner, and a_+ goes on the subdiagonal and the upper right corner. The vector $V^n = 0$ is not present due to the periodic boundary conditions. The matrix T_μ is a *circulant matrix.* The initial condition is

$$U_j^0 = \phi_0(x_j).$$

Putting $\mu = 1/a$ implies

$$u_j^{n+1} = (u_{j+1}^n + u_{j-1}^n)/2 - (u_{j+1}^n - u_{j-1}^n)/2 = u_{j-1}^n$$

which is an *upwind* method. We can write it in matrix–vector from as

$$U^{n+1} = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & & \ddots & \\ & & 1 & 0 \end{pmatrix} U^n,$$

where we see that $T_{1/a}$ is a *cyclic permutation matrix.* For this value, the method solves the scalar advection equation exactly, as it merely transports information along the characteristics.

LYCKA TILL — GOOD LUCK! G.S.

Extenta 09/12/18 (3).

1. $y_{n+1} - y_{n-1} = 2\Delta t \cdot f(y_n)$

a) $\Delta t = t_{n+1} - t_n$.

$\Rightarrow y_{n+1} - y_{n-1} = 2(t_{n+1} - t_n) \cdot f(y_n)$.

We set $y = t^m$ $f = mt^{m-1}$ $t_{n+1} = 2h$ $t_n = h$ $t_{n-1} = 0$.
$\Rightarrow \Delta t = t_{n+1} - t_n = h$

for $m = 0$ $y = 1$ and $f = 0$

$\Rightarrow 1 - 1 = 2 \cdot h \cdot 0 = 0$ Holds for $m = 0$

$m = 1$) $y = t$ $f = 1$.

$2h - 0 = 2h \cdot 1$ Holds for $m = 1$.

$m = 2$: $y = t^2$ $f = 2t$

$\underbrace{(2h)^2}_{=4h^2} - 0^2 = \underbrace{2h \cdot 2h}_{=4h^2}$ Holds for $m = 2$

$m = 3$ $y = t^3$ $f = 3t^2$

$\underbrace{(2h)^3}_{=8h^3} - 0^3 \neq \underbrace{2 \cdot h \cdot 3 \cdot h^2}_{=6h^3}$

The order of consistency is $p = 2$.

b) The Characteristic equation reads

$z^2 - 1 = 0 \Rightarrow \begin{cases} z_1 = -1 \\ z_2 = 1 \end{cases}$ a. Simple roots and $|z| \leq 1$ which fulfills the root condition so the method is zero-stable!

c) $y' = \lambda y$.

$y_{n+1} - y_{n-1} - 2\Delta t \lambda y = 0$.

The characteristic equation reads:

$z^2 - 2\Delta t \lambda z - 1 = 0$. (1)

$(z - z_1)(z - z_2) = 0$ (*) Compare quadequate characteristic equation to the one of the problem!

$z^2 - zz_2 - zz_1 + z_1z_2 = 0$.

$z^2 - (z_1 + z_2)z + z_1z_2 = 0$ Comparing this to (1) gives that

$z_1z_2 = -1$. Thus if one has modulus < 1 the other one must be > 1. Therefore they both must be 1.

2

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 1/4 & 3/4 & 0 \\ \hline & \alpha & 1-2\alpha & \alpha \end{array}$$

$h\dot{Y}_1 = hf(t_n, y_n)$

$h\dot{Y}_2 = hf(t_n + h, y_n + \frac{1}{2}\dot{Y}_1)$

$h\dot{Y}_3 = hf(t_n + h, y_n + \frac{1}{4}\dot{Y}_1 + \frac{3}{4}\dot{Y}_2) =$

$y_{n+1} = y_n + \alpha\dot{Y}_1 + (1-2\alpha)\dot{Y}_2 + \alpha\dot{Y}_3$

With the test equation $\dot{y} = \lambda y$ we get:

$h\dot{Y}_1 = h\lambda$

$h\dot{Y}_2 = h\lambda\left(1 + \frac{h\lambda}{2}\right)$

$h\dot{Y}_3 = h\lambda\left(1 + \frac{h\lambda}{4} + \frac{3}{4}h\lambda\left(1 + \frac{h\lambda}{2}\right)\right) = h\lambda\left(1 + \frac{h\lambda}{4} + \frac{3h\lambda}{4} + \frac{3(h\lambda)^2}{8}\right)$

Inserting this result into the updated step we get:

$y_1 = 1 + \alpha h\lambda + (1-2\alpha)h\lambda\left(1 + \frac{h\lambda}{2}\right) + \alpha h\lambda\left(1 + \frac{h\lambda}{4} + \frac{3h\lambda}{4} + \frac{3(h\lambda)^2}{8}\right) =$

$= 1 + \alpha h\lambda + (h\lambda - 2\alpha h\lambda)\left(1 + \frac{h\lambda}{2}\right) + \alpha h\lambda + \alpha(h\lambda)^2 + \frac{3\alpha}{8}(h\lambda)^3 =$

$= 1 + \alpha h\lambda + h\lambda - 2\alpha h\lambda + \frac{(h\lambda)^2}{2} - \alpha(h\lambda)^2 + \alpha h\lambda + \alpha(h\lambda)^2 + \frac{3\alpha}{8}(h\lambda)^3 =$

$= 1 + h\lambda(\cancel{\alpha} + 1 - \cancel{2\alpha} + \cancel{\alpha}) + (h\lambda)^2\left(\frac{1}{2} - \cancel{\alpha} + \cancel{\alpha}\right) + (h\lambda)^3\frac{3\alpha}{8} =$

$= 1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{3\alpha(h\lambda)^3}{8} = R(h\lambda)$

b) The last term needs to be $\frac{(h\lambda)^3}{3!} = \frac{(h\lambda)^3}{6} \Rightarrow$

$\Rightarrow 3\alpha\frac{(h\lambda)^3}{8} = \frac{(h\lambda)^3}{6} \Rightarrow 3\alpha = \frac{(h\lambda)^3}{6}\cdot\frac{8}{(h\lambda)^3} \Rightarrow \alpha = \frac{8}{18} = \frac{4}{9}$

c) The stability function is a polynomial as the method is explicit $\Rightarrow$ not A-stable. (Polynomial $R(h\lambda)$ and not rational $\Rightarrow$ explicit!)

3. $u'' = \lambda u \qquad u(0) = 0 \qquad u'(1) = 0$

Good Student

$N \times N$

$T_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & \cdots & 0 \\ 1 & -2 & \cdots & \\ & & & \\ & & \cdots & -1 \end{pmatrix} \Leftarrow$

$\Delta x = \frac{1}{N + \frac{1}{2}}$

Bad Student

$N \times N$

$\Delta x = \frac{1}{N+1}$

Ugly student

$(N+1) \times (N+1)$

$T_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & \cdots & 0 \\ 1 & -2 & 1 & \cdots \\ & & & \\ & & 2 & -2 \end{pmatrix}$

$\Delta x = \frac{1}{N+1}$

a) The middle graph belongs to the bad student, as it is first order as he didn't take note of the Neumann condition.

~~The other two can't be told.~~ Fuck these questions!!

2) 0 0 0

4. $-u'' - \frac{u'}{x} = \lambda u \quad u'(0)=0,\ u(R)=0$

We have a Neumann condition to the left endpoint so we set the grid $\Delta x = \frac{R}{N+\frac{1}{2}}$ with N interior grid points and $x_k = \frac{\Delta x}{2} + (k-1)\Delta x$

Bessel function with division of $\frac{1}{x_k}$ ($\frac{1}{2}$). Need to somehow have a point at zero!

The Neumann condition $u'(1)=0$ is approximated by

$$\frac{u_1 - u_0}{\Delta x} = 0 \Rightarrow u_0 = u_1$$

Discretization:

$$\frac{u_{n-1} - 2u_n + u_{n+1}}{\Delta x^2} - \frac{u_{n+1} + u_{n-1}}{2x_n \Delta x} = \lambda_{\Delta x} u_n.$$

So with BC:

$$-\frac{-2u_1 + 2u_2}{\Delta x^2} = \lambda_{\Delta x} u_1$$

$$-\frac{u_{k-1} - 2u_k + u_{k+1}}{\Delta x^2} - \frac{-u_{k-1} + u_{k+1}}{2x_k \Delta x} = \lambda_{\Delta x} u_k \qquad k = 2 : N-1$$

$$(2k-1)\Delta x^2$$

$$-\frac{u_{N-1} - 2u_N}{\Delta x^2} - \frac{-u_{N-1}}{2x_N \Delta x} = \lambda_{\Delta x} u_N$$

$$(2N-1)\Delta x^2$$

$$2x_k = \frac{\Delta x}{2} + (k-1)\Delta x = \left(\frac{1}{2} + k - 1\right)\Delta x = \left(k - \frac{1}{2}\right)\Delta x$$

$$\Rightarrow x_k = (2k-1)\Delta x.$$

So for $A_{\Delta x} u = \lambda_{\Delta x} u$

$$A_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} 2 & -2 & 0 & \\ -1+\frac{1}{2k-1} & 2 & -1-\frac{1}{2k-1} & \\ & & \ddots & \\ & & -\frac{2N+2}{2N+1} & 2 \end{pmatrix}$$

5 a) $u_t = u_{xx} + f(u)$. Parabolic

CFL: $\frac{\Delta t}{\Delta x^2}$

b) $u_t + u_x = 0$ Hyperbolic

CFL: $\frac{\Delta t}{\Delta x}$

c) $u_t + u_x = u_{xx}$ Parabolic

CFL = $\frac{\Delta t}{\Delta x^2}$

d) $u_t + uu_x = u_{xx}$ Parabolic

CFL: $\frac{\Delta t}{\Delta x^2}$

Journal on thursday / friday.

e) $u_{tt} = u_{xx}$ Hyperbolic

CFL: $\frac{\Delta t}{\Delta x}$

6. $u_i^{n+1} = (u_{i+1}^n + u_{i-1}^n)/2 - a\Delta t\,(u_{i+1}^n - u_{i-1}^n)/2\Delta x$

a) $\mu = \frac{\Delta t}{\Delta x}$) $u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{a\mu}{2}(u_{i+1}^n - u_{i-1}^n)$

$U^{n+1} = T_\mu U^n + V^n \Rightarrow$

$$\Rightarrow T_\mu = \begin{pmatrix} 0 & \frac{1-a\mu}{2} & \cdot & \frac{1+a\mu}{2} \\ \frac{1+a\mu}{2} & 0 & \frac{1-a\mu}{2} & \vdots \\ \frac{1-a\mu}{2} & \cdots & \frac{1+a\mu}{2} & 0 \end{pmatrix}$$

$V^n = 0$.

$U^n = [\phi(x_1), \phi(x_2), \dots \phi(x_j)]^T$.

Matrix is circulant (periodic bc).

b) $\mu = \frac{1}{a}$ $\frac{1 - a\cdot\frac{1}{a}}{2} = 0$. $\frac{1 + a\cdot\frac{1}{a}}{2} = 1$. Upwind VS Downwind

upwind yeah!!

$$u_i^{n+1} = \frac{u_{i+1} + u_{i-1}}{2} - \frac{u_{i+1} - u_{i-1}}{2} = u_{i-1}^n$$

$$T_{1/a} = \begin{pmatrix} 0 & 0 & & 1 \\ 1 & 0 & 0 & \vdots \\ & & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

Since it uses points from the left as it goes to the right it's an upwind method!

Extenta #1.
08/12/18

$\dot{y} = f(y)$

leapfrog midpoint method $y_{n+1} - y_{n-1} = 2\Delta t \cdot f(y_n)$

$y(t) = t^p$ $\quad f(y) = p t^{p-1}$

$$\begin{cases} t_{n-1} = -\Delta t \\ t_n = 0 \\ t_{n+1} = \Delta t \end{cases}$$

p	y_{n+1}	y_{n-1}	$f(y_n)$	formula	
0	$(t_{n+1})^0 = (\Delta t)^0 = 1$	$(t_{n-1})^0 = (-\Delta t)^0 = 1$	$0 \cdot (t_n)^{-1} = 0$	$1 - 1 = 2\Delta t \cdot 0$	✓
1	$(\Delta t)^1 = \Delta t$	$(-\Delta t)^1 = -\Delta t$	$1 \cdot (0)^0 = 1$	$\Delta t - (-\Delta t) = 2\Delta t \cdot 1$	✓
2	$(\Delta t)^2 = \Delta t^2$	$(-\Delta t)^2 = \Delta t^2$	$2 \cdot (0)^1 = 0$	$\Delta t^2 - \Delta t^2 = 2\Delta t \cdot 0$	✓
3	$(\Delta t)^3 = \Delta t^3$	$(-\Delta t)^3 = -\Delta t^3$	$3 \cdot (0)^3 = 0$	$\Delta t^3 - (-\Delta t^3) = 2\Delta t \cdot 0$	✗

The method's order of consistency is 2.

$f(y) = 0$ Characteristic equation: $z^2 - 1 = 0$

$\Rightarrow z = \pm 1$ and $|z| \leq 1$ means that the method zero-stable.

$\dot{y} = \lambda y$

$y_{n+1} - 2\Delta t \lambda y_n - y_{n-1} = 0$

this gives the characteristic equation

$z^2 - 2\Delta t \lambda z - 1 = 0$

$(z - z_1)(z - z_2) = z^2 - z(z_1 + z_2) + z_1 z_2 = 0 \Rightarrow z_1 z_2 = -1.$

$|z_1| \cdot |z_2| = 1 \Rightarrow$ if $|z_1| < 1$ $|z_2| > 1$ and vice versa meaning that both must have modulus 1 to be stable.

2

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 1/4 & 3/4 & 0 \\ \hline & \alpha & 1-2\alpha & \alpha \end{array}$$

a) Find its stability function $R(h\lambda)$

$h\cdot\dot{Y}_1 = h\lambda\cdot y_n$

$h\cdot\dot{Y}_2 = h\lambda\cdot(y_n + h\dot{Y}_1/2) = h\lambda(1 + h\lambda/2)\cdot y_n$

$h\cdot\dot{Y}_3 = h\lambda\cdot(y_n + h\dot{Y}_1/4 + 3h\dot{Y}_2/4) =$

$= h\lambda(1 + h\lambda/4 + 3h\lambda(1+h\lambda/2)/4)\cdot y_n =$

$= h\lambda(1 + h\lambda/4 + 3h\lambda/4 + 3(h\lambda)^2/8)\, y_n$

$y_{n+1} = y_n + \alpha(h\dot{Y}_1 + h\dot{Y}_3) + (1-2\alpha)h\dot{Y}_2 =$

$= y_n + \alpha(h\lambda + h\lambda(1 + h\lambda + 3(h\lambda)^2/8))\, y_n + (1-2\alpha)h\lambda(1+h\lambda/$

$= y_n(1 + \alpha h\lambda + \alpha h\lambda(1 + h\lambda + 3(h\lambda)^2/8) + (1-2\alpha)(h\lambda + (h\lambda)^2/$

$= y_n(1 + \cancel{\alpha h\lambda} + \cancel{\alpha h\lambda} + \alpha(h\lambda)^2 + 3\alpha(h\lambda)^3/8 + h\lambda + (h\lambda)^2/2 - 2\alpha$

$= y_n(1 + \alpha(\cancel{h\lambda + h\lambda} + \cancel{(h\lambda)^2} + 3(h\lambda)^3/8 - \cancel{2h\lambda} - \cancel{(h\lambda)^2}$

$= y_n(1 + h\lambda + (h\lambda)^2/2 + 3\alpha(h\lambda)^3/8)$

Thus: $R(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{3\alpha(h\lambda)^3}{8}$

b) $\frac{4}{9} \Rightarrow R(h\lambda) = \dots \frac{(h\lambda)^3}{6}$

c) No, the method is explicit.

The middle is of order 1 - the bad student.

There is no way to know the other two.

He used the standard grid $\Delta x = \frac{1}{N+1}$

s means that he needs to solve for u_{N+1} at $x_{N+1} = 1$
ich implies an extra point and therefore $(N+1) \times (N+1)$
trix.
then represented the Neuman condition to 2nd
der and inserted it to the last equation and thus
odifying the last row.

$-u'' - \frac{u'}{x} = \lambda u \qquad u'(0),\ u(R) = 0$

$\frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2} - \frac{1}{x} \cdot \frac{u_{n+1} + u_{n-1}}{2\Delta x} = \lambda u$

rid Points: $\Delta x = \frac{R}{N + 1/2}$ ~~while 'inner' are $\Delta x_2 \to \Delta x_{N+1}$~~

$x_0 = 0 \qquad x_{N+1} = R$

$x_k = \frac{\Delta x}{2} + (k-1)\Delta x \qquad k = 1:N$

s gives discretization:

$\frac{u_2 - 2u_1 + u_0}{\Delta x^2} - \frac{1}{x_1} \frac{u_2 - u_0}{2\Delta x} = \lambda_{\Delta x} u_1$

$\frac{u_{k+1} - 2u_k + u_{k-1}}{\Delta x^2} - \frac{1}{x_k} \frac{u_{k+1} - u_{k-1}}{2\Delta x} = \lambda_{\Delta x} u_k$

$\frac{u_{N+1} - 2u_N + u_{N-1}}{\Delta x^2} - \frac{1}{x_N} \frac{u_{N+1} - u_{N-1}}{2\Delta x} = \lambda_{\Delta x} u_N$

undary cond: $u_{N+1} = 0$, $u_0 = u_1 \Rightarrow$

$\frac{2u_2 - 2u_1}{\Delta x^2} = \lambda_{\Delta x} u_1$

$\frac{u_{k+1} - 2u_k + u_{k-1}}{\Delta x^2} - \frac{1}{x_k} \frac{u_{k+1} - u_{k-1}}{2\Delta x} = \lambda_{\Delta x} u_k$

$\frac{-2u_N + u_{N-1}}{\Delta x^2} - \frac{1}{x_N} \frac{-u_{N-1}}{2\Delta x} = \lambda_{\Delta x} u_N$

5. $t \geq 0 \quad x \in [0,1]$

a) $u_t = u_{xx} + f(u)$; Parabolic, $p=2$. {REACTION DIFF}

b) $u_t + u_x = 0$; Hyperbolic, $p=1$ {ADVECTION}

c) $u_t + u_x = u_{xx}$; Parabolic, $p=2$ {CONV - DIFF}

d) $u_t + uu_x = u_{xx}$; Parabolic, $p=2$ {VISCOUS BURGERS}

e) $u_{tt} = u_{xx}$; HYPERBOLIC; $p=1$ {Wave}

6. $u_t + au_x = 0 \quad 0 \leq x \leq 1, \; t \geq 0$
$u(x,0) = \phi_0(x)$.

$$u_j^{n+1} = (u_{j+1}^n + u_{j-1}^n)/2 - a\Delta t\,(u_{j+1}^n - u_{j-1}^n)/2\Delta x$$

a) $\mu = \Delta t/\Delta x$

$a_+ = (1 + a\mu)/2 \qquad a_- = (1 - a\mu)/2$.

$$U^{n+1} = \begin{pmatrix} 0 & a_- & \cdots & a_+ \\ a_+ & 0 & a_- & \vdots \\ & & \ddots & \\ a_- & \cdots & a_+ & 0 \end{pmatrix} U^n = T_\mu U^n$$

Circulant matrix.

b) $\mu = \frac{1}{a} \Rightarrow (u_{j+1}^n + u_{j-1}^n)/2 - (u_{j+1}^n - u_{j-1}^n)/2 = u_{j-1}^n$

$$U^{n+1} = \begin{pmatrix} 0 & & & 1 \\ & 0 & & \vdots \\ & & \ddots & \\ & & 1 & 0 \end{pmatrix} U^n \text{ is an upwind method.}$$

EXTENTA #2

Numerical Methods for Differential Equations FMNN10/NUMN12
Final exam 2010-12-14 **Grades to be announced 2010-12-21**
Solution sketch by Gustaf Söderlind

Exam duration 08:00 – 13:00. A minimum of 15 points out of 30 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, textbooks, lecture notes or any other electronic or written material may be used during the exam.

1. **(5p)** For the initial value problem $\dot{y} = f(y)$ the *2-step BDF method* reads
$$\left(\nabla + \frac{\nabla^2}{2}\right) y_n = h \cdot f(y_n),$$
where ∇ is the backward difference operator.

   (a) Find the method's order of consistency. (2p)

   (b) Show that the method is zero-stable. (1p)

   (c) Construct the nonlinear equation that has to be solved on each step to use the method. What method should be used to solve this equation? (2p)

Solution. Using $\nabla y_n = y_n - y_{n-1}$ and $\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$, the method can be written
$$\frac{3}{2}y_n - 2y_{n-1} + \frac{1}{2}y_{n-2} = h \cdot f(y_n).$$

To find the order of consistency, we insert the polynomials t^m for $m = 0, 1, 2, \ldots$. Starting with $P(t) = 1$, $P'(t) = 0$, we get
$$\text{LHS} = \frac{3}{2} - 2 + \frac{1}{2} = 0; \quad \text{RHS} = h \cdot 0 = 0,$$
implying that the formula holds for $m = 0$. For $P(t) = t^m$, with $P'(t) = mt^{m-1}$, we get
$$\begin{aligned}\text{LHS} &= \frac{3}{2} \cdot (2h)^m - 2 \cdot h^m + \frac{1}{2} \cdot 0^m = (3 \cdot 2^{m-1} - 2)h^m \\ \text{RHS} &= h \cdot m(2h)^{m-1} = m2^{m-1}h^m,\end{aligned}$$
so LHS equals RHS for $m = 1$ and $m = 2$, but fails for $m = 3$. Therefore the order of consistency is $p = 2$.

For zero-stability, we consider the characteristic equation
$$\frac{3}{2}z^2 - 2z + \frac{1}{2} = 0.$$

The characteristic polynomial can be factorized

$$\frac{3}{2}(z-1)(z-\frac{1}{3}),$$

showing that apart from the mandatory unit root, the only other root is $1/3$, inside the unit circle. Hence the root condition is satisfied and the method is zero-stable.

In advancing the solution one step, given y_{n-1} and y_{n-2}, we have to solve the (nonlinear) equation

$$\frac{3}{2}y_n - hf(y_n) = 2y_{n-1} - \frac{1}{2}y_{n-2},$$

for the unknown y_n (present in the left-hand side), while the data (known, past values) appear in the right-hand side. The equation should be solved using Newton's method, as the BDF methods are intended for stiff differential equations.

2. **(5p)** Consider the 2-stage implicit Runge-Kutta method with Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1-\theta & \theta \\ \hline & 1-\theta & \theta \end{array}$$

   (a) Find its stability function $R(h\lambda)$. (3p)

   (b) For what values of θ is the method A-stable? (2p)

Solution. We apply the method to the linear test equation $y' = \lambda y$ with initial condition $y_0 = 1$. We get

$$\begin{aligned} hY_1' &= h\lambda \\ hY_2' &= h\lambda \cdot \left(1 + (1-\theta)hY_1' + \theta hY_2'\right) \end{aligned}$$

Here we have to solve $(1-\theta h\lambda) \cdot hY_2' = h\lambda \cdot (1 + (1-\theta) \cdot h\lambda)$ for the (scaled) stage derivative hY_2', to get

$$hY_2' = \frac{h\lambda \cdot (1 + (1-\theta) \cdot h\lambda)}{1-\theta h\lambda}.$$

Therefore,

$$\begin{aligned} y_1 &= 1 + (1-\theta) \cdot hY_1' + \theta \cdot hY_2' \\ &= \frac{(1-\theta h\lambda) + (1-\theta h\lambda)(1-\theta)h\lambda + \theta h\lambda \cdot (1 + (1-\theta) \cdot h\lambda)}{1-\theta h\lambda} \\ &= \frac{1 + (1-\theta)h\lambda}{1-\theta h\lambda} = R(h\lambda). \end{aligned}$$

This is the θ method. For $\theta = 0$ we have the explicit Euler method (not A-stable); for $\theta = 1/2$ the Trapezoidal rule (A-stable); and for $\theta = 1$ we have the implicit Euler method (also A-stable).

When $0 < \theta < 1$, for large $h\lambda$ we have

$$|R(h\lambda)| \to |\frac{(1-\theta)h\lambda}{-\theta h\lambda}| = \frac{1-\theta}{\theta},$$

which is less than or equal to 1 in magnitude if and only if $1/2 \leq \theta \leq 1$. Hence the method is A-stable for $1/2 \leq \theta \leq 1$.

3. **(6p)** The three students, **the Good**, **the Bad** and **the Ugly**, are back in the computer lab this year too, now trying to solve the beam equation, $M'' = q$; $u'' = M/(EI)$, with boundary conditions $M(0) = M(10) = 0$ and $u(0) = u(10) = 0$, using three different methods.

   In order to solve any equation of the form $y'' = f$ with homogeneous boundary conditions, they all use methods of the form $T_{\Delta x}y = Bf$, where the $N \times N$ matrix $T_{\Delta x}$ is given by

$$T_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & \dots & 0 \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -2 \end{pmatrix}$$

   and $\Delta x = 10/(N+1)$. They cannot agree on the matrix B, however.

   The **Good Student** uses the Finite Difference Method (FDM) and takes $B_{\text{FDM}} = I$, the identity matrix. The **Bad Student**, on the other hand, prefers the Finite Element Method (FEM) with piecewise linear elements, and takes

$$B_{\text{FEM}} = \frac{1}{6}\begin{pmatrix} 4 & 1 & \dots & 0 \\ 1 & 4 & 1 & \\ & & \ddots & \\ & & 1 & 4 \end{pmatrix}.$$

   While they are disputing the advantages and disadvantages of their methods, the **Ugly Student** unexpectedly enters, and says that both FDM and FEM are 2nd order methods, and that if one is going to use the B matrix at all, one had better use it to increase the order of the method. Keeping it a secret that he has found an interesting 4th order method known as *Cowell's method* in a book, the Ugly student smiles mysteriously and tells the Good and the Bad that according to Aristotle, the best is usually found between the extremes. Thus he

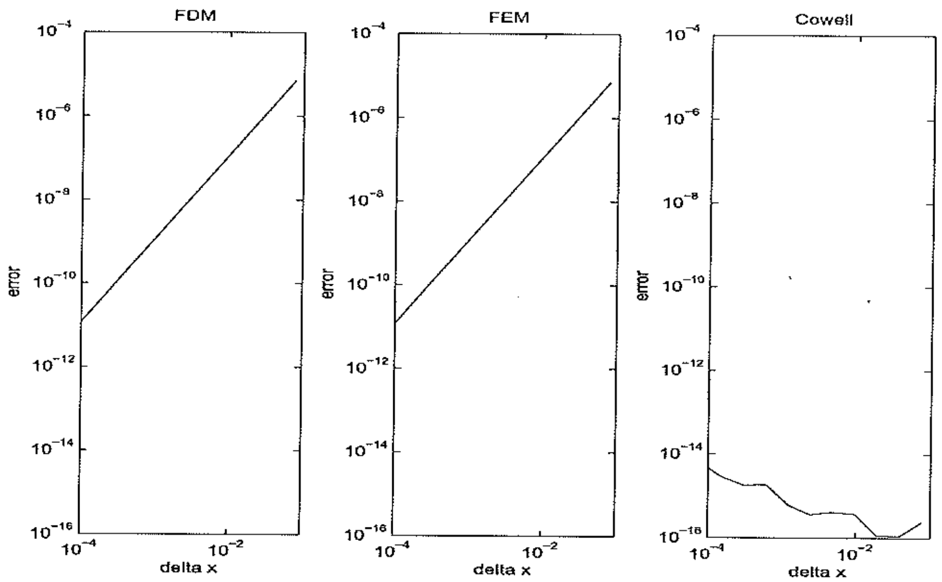


Figure 1: Error, as a function of Δx, in the computation of the deflection of a beam for three different methods, devised by three different students

proposes a compromise, instead choosing B as follows:

$$B_{\text{Cowell}} = \frac{1}{2}(B_{\text{FDM}} + B_{\text{FEM}}) = \frac{1}{12}\begin{pmatrix} 10 & 1 & \dots & 0 \\ 1 & 10 & 1 & \\ & & \ddots & \\ & & 1 & 10 \end{pmatrix}.$$

This means that the problem $y'' = f$ is discretized as

$$\frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} = \frac{f_{n-1} + 10f_n + f_{n+1}}{12}. \tag{1}$$

The Ugly claims that this will produce far superior accuracy. As the Good and the Bad don't believe him, they all agree on a simple test problem with q constant, with all other parameters E and I also constant, and find the analytical solution

$$u(x) = \frac{q}{24EI}\left(1000x - 20x^3 + x^4\right).$$

They decide to compute and plot the error for different Δx for their respective methods. The graphs of the error when the three students' codes were run are shown in Figure 1.

(a) Show that the Ugly student's method is at least $p = 4$ by investigating the discretization (1). (4p)

(b) Explain the unusual behavior of the Ugly student's error graph (see Figure 1) and why one *doesn't* observe a 4th order slope, contrary to expectations. (2p)

Solution. In (1), we insert the exact solution and expand in Taylor series around x_n to get

$$y(x_{n-1}) = y - \Delta x y' + \frac{\Delta x^2}{2!}y'' - \frac{\Delta x^3}{3!}y^{(3)} + \frac{\Delta x^4}{4!}y^{(4)} - \frac{\Delta x^5}{5!}y^{(5)} + \mathrm{O}(\Delta x^6)$$
$$y(x_{n+1}) = y + \Delta x y' + \frac{\Delta x^2}{2!}y'' + \frac{\Delta x^3}{3!}y^{(3)} + \frac{\Delta x^4}{4!}y^{(4)} + \frac{\Delta x^5}{5!}y^{(5)} + \mathrm{O}(\Delta x^6).$$

Therefore

$$\frac{y(x_{n-1}) - 2y(x_n) + y(x_{n+1})}{\Delta x^2} = y''(x_n) + \frac{\Delta x^2}{12}y^{(4)} + \mathrm{O}(\Delta x^4).$$

But the right-hand side also needs to be expanded. Noting that $y'' = f$ (and therefore $y^{(4)} = f''$), we get

$$\begin{aligned}\frac{f(x_{n-1}) + 10f(x_n) + f(x_{n+1})}{12} &= f(x_n) + \frac{f(x_{n-1}) - 2f(x_n) + f(x_{n+1})}{12} \\ &= y''(x_n) + \frac{\Delta x^2}{12}y^{(4)} + \mathrm{O}(\Delta x^4),\end{aligned}$$

so the left and right-hand sides of (1) differ by at most $\mathrm{O}(\Delta x^4)$, showing that the method's order of consistency is at least $p = 4$. (In fact, it *is* $p = 4$.)

As a simpler alternative, one can verify that the discretization formula holds exactly for polynomials up to degree 4.

The unusual behavior of the error graph is due to the fact that the test problem's solution is a polynomial of degree 4. Being of order $p = 4$, Cowell's method solves all such problems exactly, leaving only round-off errors $\sim 10^{-15}$. A 4th order slope will only be observed for more general problems.

4. **(4p)** A special type of Sturm–Liouville eigenvalue problems has the form

$$y'' + d(x)y = \lambda y$$

with boundary conditions $y'(0) = y(1) = 0$, and the function $d(x) > 0$ on $[0, 1]$. Formulate this as an algebraic eigenvalue problem

$$Ay = \lambda_{\Delta x} y.$$

by using a *second order discretization.* Take care to define the grid, the mesh-width Δx, and construct the matrix A. Don't forget to define the dimensions of the matrix.

Solution. We take $\Delta x = 1/(N+1/2)$ to take care of the Neumann boundary condition on the left, which is approximated by

$$\frac{y_1 - y_0}{\Delta x} = 0,$$

implying that $y_0 = y_1$. For $n = 1, \ldots, N$ our discrete eigenvalue problem is

$$\frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} + d(x_n)y_n = \lambda_{\Delta x} y_n.$$

Introduce the diagonal matrix $D = \text{diag}(d(x_1)\ d(x_2)\ \ldots\ d(x_N))$. The algebraic eigenvalue problem can then be written

$$(T_{\Delta x} + D)y = \lambda_{\Delta x} y,$$

where the $N \times N$ matrix $T_{\Delta x}$ is given by

$$T_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} -1 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -2 \end{pmatrix}.$$

Here the top left matrix element accounts for the Neumann condition $y'(0) = 0$. So the matrix $A = T_{\Delta x} + D$. This is a simple diagonal modification to the usual matrix for $y'' = \lambda y$.

5. **(5p)** For $t \geq 0$ and $x \in [0,1]$, let u_j^n approximate $u(j \cdot \Delta x, n \cdot \Delta t)$. Below you find some simple discretizations of some important prototypical PDEs. Write down the differential equations corresponding to the discretizations below, and give the name of the equation in each case.

(a)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + f(u_j^n)$$

(b)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n + u_j^n}{2} \cdot \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

(c)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

(d)

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{u_j^n-u_{j-1}^n}{\Delta x}=0$$

(e)

$$\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}$$

Solution.

(a) $u_t = u_{xx} + f(u)$ Reaction-diffusion equation

(b) $u_t + uu_x = 0$ (Inviscid) Burgers equation

(c) $u_t = u_{xx} + u_x$ Convection-diffusion equation

(d) $u_t + u_x = 0$ Advection equation

(e) $u_{tt} = u_{xx}$ Wave equation

6. **(5p)** The Lax–Friedrichs scheme for the advection problem with $a > 0$ and periodic boundary conditions,

$$\begin{aligned} &u_t + au_x = 0, \quad 0 \le x \le 1, \quad t \ge 0, \\ &u(x,0) = \phi_0(x), \end{aligned}$$

is

$$u_j^{n+1} = \frac{u_{j+1}^n + u_{j-1}^n}{2} - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n).$$

(a) Let $\mu = \Delta t/\Delta x$. Rewrite this scheme as a matrix–vector recursion $U^{n+1} = T_\mu U^n$ with $U^n = [u_1^n, u_2^n, \ldots, u_N^n]^{\mathrm{T}}$, giving the matrix T_μ and the vector U^0, taking care to define Δx in accordance with matrix dimensions. Is the matrix symmetric, skew-symmetric, nonsymmetric, circulant, Toeplitz, or of some other type? (3p)

(b) Put $\mu = 1/a$ in the matrix. Calculate its eigenvalues and conclude that the method is stable. (2p)

Solution. The $N \times N$ matrix T_μ is

$$T_\mu = \frac{1}{2}\begin{pmatrix} 0 & 1-a\mu & 0 & & & 1+a\mu \\ 1+a\mu & 0 & 1-a\mu & & & \\ & & \ddots & & & \\ & & & & & \\ 1-a\mu & & & 0 & 1+a\mu & 0 \end{pmatrix}.$$

It is a circulant matrix (due to the periodic boundary conditions), and also a Toeplitz matrix. The vector $U^0 = [\phi(x_1), \phi(x_2), \ldots, \phi(x_N)]^{\mathrm{T}}$, with $x_j = j/N$.

In case $a\mu = 1$ we get the matrix

$$T_{1/a} = \begin{pmatrix} 0 & 0 & & & 1 \\ 1 & 0 & & & \\ & & & \ddots & \\ 0 & & 0 & 1 & 0 \end{pmatrix}.$$

This is a cyclic permutation matrix, with $T_{1/a}^N = I$. (After N steps, the wave has come back to its original position.)
If $T_{1/a}$ has eigenvalues λ, then $T_{1/a}^N$ has eigenvalues λ^N. But since $T_{1/a}^N = I$, we have $\lambda^N = 1$, implying

$$\lambda_k = \mathrm{e}^{2\pi \mathrm{i} k/N}; \qquad k = 1, \ldots, N.$$

Because all eigenvalues have unit modulus and are simple, the method is stable.

Extenta 2010-12-14 ②

1 $\left(\nabla + \frac{\nabla^2}{2}\right) y_n = hf(y_n)$

a) $\nabla y_n = y_n - y_{n-1}$ and $\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$ gives:

$y_n - y_{n-1} + \frac{1}{2} y_n - y_{n-1} + \frac{1}{2} y_{n-2} = hf_n$

$\Rightarrow \frac{3}{2} y_n - 2 y_{n-1} + \frac{1}{2} y_{n-2} = hf_n$

With $y = P(t)$ and $P(t) = t^m$
$f = P'(t)$ $\quad P'(t) = mt^{m-1}$

and also $t_n = 2h$, $t_{n-1} = h$, $t_{n-2} = 0$ we investigate the order of consistency:

For $m = 0$ we have $P(t) = 1$ and $P'(t) = 0$

$\frac{3}{2} \cdot 1 - 2 \cdot 1 + \frac{1}{2} \cdot 1 = 0 \Leftrightarrow \frac{3}{2} - \frac{3}{2} = 0 = 0$. It holds for order $m = 0$

Order $m = 1$:

$\frac{3}{2} \cdot (2h)^1 - 2 \cdot h^1 + \frac{1}{2} \cdot 0^1 = h \cdot 1 \cdot (2h)^0 \Rightarrow$

$\Rightarrow h = h$ Holds for order $m = 1$.

Order $m = 2$.

$\frac{3}{2} \cdot (2h)^2 - 2 \cdot h^2 + \frac{1}{2} \cdot 0 = h \cdot 2 \cdot 2h$

$\Rightarrow 6h^2 - 2h^2 = 4h^2 = 4h^2$ Holds for order $m = 2$

Order $m = 3$

$\frac{3}{2} \cdot (2h)^3 - 2h^3 + \frac{1}{2} \cdot 0 = h \cdot 3 \cdot (2h)^2$

$12h^3 - 2h^3 = 10h^3 \neq 12h^3$ Doesn't hold for order $m = 3$

$\Rightarrow$ The order of consistency is $p = 2$.

b) For zero-stability it must fulfill the root condition.

The characteristic equation reads:

$\frac{3}{2} z^2 - 2z + \frac{1}{2} = 0 \Rightarrow z^2 - \frac{4}{3} z + \frac{1}{3} = 0$.

$\Rightarrow \left(z - \frac{2}{3}\right)^2 + \frac{1}{3} - \frac{4}{9} = 0 \Rightarrow \left(z - \frac{2}{3}\right)^2 = \frac{1}{9} \Rightarrow z = \frac{2}{3} \pm \frac{1}{3} \Rightarrow \begin{cases} |z_1| = 1 \\ |z_2| = \frac{1}{3} \end{cases}$

We have simple unit moduli and $|z_i| \leq 1$ for all (both) i $(1 \leq i \leq 2)$ so thus the method is zero-stable.

c) To construct the non-linear equation we gather the unknowns on the lhs:

$\frac{3}{2}y_n - hf_n = 2y_{n-1} - \frac{1}{2}y_{n-2}$, we here have each step on the left hand side and previously computed steps on the right hand side.

Newtons method should be used, since BDF methods are for stiff problems.

2 a)

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1-\Theta & \Theta \\ \hline & 1-\Theta & \Theta \end{array}$$

$$hY_1' = hf(t_n, y_n)$$
$$hY_2' = hf(t_n+h, y_n + (1-\Theta)hY_1' + \Theta hY_2')$$
$$y_{n+1} = y_n + (1-\Theta)Y_1' + \Theta hY_2'$$

We put $y' = \lambda y$ so that

$$hY_1' = h\lambda$$
$$hY_2' = h\lambda\left(1 + (1-\Theta)h\lambda + \Theta hY_2'\right)$$
$$\Rightarrow hY_2'(1-\Theta h\lambda) = h\lambda(1+(1-\Theta)h\lambda)$$
$$\Rightarrow hY_2' = h\lambda\frac{1+(1-\Theta)h\lambda}{1-\Theta h\lambda}$$

Inserting this into the updated step gives us:

$$y_1 = 1 + (1-\Theta)h\lambda + \Theta h\lambda\frac{1+(1-\Theta)h\lambda}{1-\Theta h\lambda} = \frac{1-\cancel{\Theta h\lambda} + (1-\Theta)h\lambda(1-\Theta h\lambda) + \cancel{\Theta h\lambda} + (1-\Theta)h\lambda\Theta h\lambda}{1-\Theta h\lambda}$$
$$= \frac{1+(1-\Theta)h\lambda(1-\cancel{\Theta h\lambda}+\cancel{\Theta h\lambda})}{1-\Theta h\lambda} = \frac{1+(1-\Theta)h\lambda}{1-\Theta h\lambda} = R(h\lambda)$$

b) For A-stability: $\begin{cases} |R(i\omega)|^2 \le 1 \\ \text{Only positive real parts for the poles} \end{cases}$

Poles: $1+(1-\Theta)i\omega = 0 \Rightarrow$ No pole in the LHP for all Θ

$|R(i\omega)|^2 = \left|\frac{1+(1-\Theta)i\omega}{1-\Theta i\omega}\right|^2 = \frac{1+(1-\Theta)^2\omega^2}{1+\Theta^2\omega^2} \le 1$ which is, when

$0 \le \Theta \le 1$ and large ω: $\frac{(1-\Theta)\cancel{\omega}}{\Theta\cancel{\omega}} = \frac{1-\Theta}{\Theta} \le 1$

$\Rightarrow 1-\Theta \le \Theta \Rightarrow 2\Theta \ge 1 \Rightarrow \Theta \ge \frac{1}{2}$ → Pretty nasty there.

Thus, for the method to be A-stable, $\frac{1}{2} \le \Theta \le 1$.

3. $M'' = q$ $\quad u'' = \frac{M}{EI}$ $\quad M(0) = M(10) = 0$

$u(0) = u(10) = 0$

$T_{\Delta x} y = Bf$ $\quad T_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{pmatrix}$ $\quad \Delta x = \frac{10}{N+1}$

Good student: $\quad$ Bad student: $\quad$ Ugly student:

$B_{FDM} = I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ $\quad B_{FEM} = \frac{1}{6}\begin{pmatrix} 4 & 1 & & 0 \\ 1 & 4 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & 4 \end{pmatrix}$ $\quad B_{Cowell} = \frac{1}{2}(B_{FDM} + B_{FEM}) = \frac{1}{12}\begin{pmatrix} 10 & 1 & & 0 \\ 1 & 10 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & 10 \end{pmatrix}$

$$\frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} = \frac{f_{n-1} + 10f_n + f_{n+1}}{12}$$

Analytical solution: $u(x) = \frac{q}{24EI}(1000x - 20x^3 + x^4)$

a) with $y = t^m$ and $f = y'' = (m^2 - m)t^{m-2}$ $\quad x_n$

$t_{n+1} = 2\Delta x$ $\quad t_n = \Delta x$ $\quad t_{n-1} = 0$

Normal polynomial test = t^p

$m = 0$ $\quad y = 1$ $\quad f = 0$

$\frac{1 - 2 + 1}{\Delta x^2} = \frac{0}{12} \Rightarrow 0 = 0$ $\quad$ Holds for $m = 0$.

$m = 1$: $\frac{0 - 2\Delta x + 2\Delta x}{\Delta x^2} = 0$ $\quad$ Holds for $m = 1$.

$y = t^1$
$f = 0$

$m = 2$: $\frac{0^2 - 2(\Delta x)^2 + (2\Delta x)^2}{\Delta x^2} = \frac{2 + 20 + 2}{12} \Rightarrow 2 = 2$ $\quad$ Holds for $m = 2$.

$y = t^2$
$f = 2$

$m = 3$: $\frac{0^3 - 2(\Delta x)^3 + (2\Delta x)^3}{\Delta x^2} = \frac{6 \cdot 0 + 60\Delta x + 12\Delta x}{12}$

$y = t^3$

$f = 6t$ $\Rightarrow \frac{-2\Delta x^3 + 8\Delta x^3}{\Delta x^2} = 6\Delta x \Rightarrow 6\Delta x = 6\Delta x$ $\quad$ Holds for $m = 3$.

$m = 4$: $\frac{0^4 - 2(\Delta x)^4 + (2\Delta x)^4}{\Delta x^2} = \frac{12 \cdot 0^2 + 120(\Delta x)^2 + 12(2\Delta x)^2}{12} \Leftrightarrow$

$y = t^4$
$f = 12t^2$

$\Rightarrow \frac{-2\Delta x^4 + 16\Delta x^4}{\Delta x^2} = \frac{120\Delta x^2 + 48\Delta x^2}{12} \Rightarrow 14\Delta x^2 = 14\Delta x^2$ $\quad$ Holds for $m = 4$!

$\therefore$ The method's order of consistency is at least $p = 4$!

b) The test problem is of order $p=4$ and Cowells method solves those problems exactly leaving roundoff errors.

Same order problem
same solution
roundoff errors!

4. $y'' + d(x)\, y = \lambda y$ $\quad y'(0) = y(1) = 0 \quad d(x) > 0$ on $[0,1]$

$$A y = \lambda_{\Delta x}\, y$$

We have a Neumann condition at the left endpoint, so $\Delta x = \frac{1}{N+\frac{1}{2}}$ with N interior grid points and $x_1 = 1 \cdot \Delta x$.

The Neumann condition $y'(0) = 0$ is approximated by

$\frac{y_1 - y_0}{\Delta x} = 0 \Rightarrow y_1 = y_0$. The discretization of our problem is $\frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} + d(x_i)\, y_i = \lambda_{\Delta x} y_i$

So we have:

$$\frac{-y_1 + y_2}{\Delta x^2} + d(x_1) y_1 = \lambda_{\Delta x} y_1$$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} + d(x_i) y_i = \lambda_{\Delta x} y_i \qquad i = 2 : N-1$$

$$\frac{y_{N-1} - 2y_N}{\Delta x^2} + d(x_N) y_N = \lambda_{\Delta x} y_N.$$

Thus we have.

$$A = \frac{1}{\Delta x^2}\begin{pmatrix} -1+d(x_1) & 1 & & 0 \\ 1 & -2+d(x_2) & 1 & \\ & & 1 & -2+d(x_N) \\ & 1 & & \end{pmatrix}$$

Further we can write $A = (T_{\Delta x} + D)$ wher $T_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ & 1 & -2 \end{pmatrix}$

and $D = \mathrm{diag}(d(x_1), d(x_2) \dots d(x_N))$.

This is a $N \times N$ matrix!

5 a) $u_t = u_{xx} + f(u)$ Reaction-Diffusion equation.

b) $u_t + uu_x = 0$ Inviscid Burger

c) $u_t = u_{xx} + u_x$ Convection diffusion

d) $u_t + u_x = 0$ Advection equation

e) $u_{tt} = u_{xx}$ Wave equation

6. $u_t + au_x = 0 \qquad 0 \le x < 1 \qquad t \ge 0$

$u(x,0) = \phi_0(x)$

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

a) $\mu = \frac{\Delta t}{\Delta x} \qquad U^{n+1} = T_\mu U^n$

We have $x_i = \frac{i}{N} \Rightarrow \Delta x = \frac{1}{N}$

We insert μ:

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{a\mu}{2}(u_{i+1}^n - u_{i-1}^n)$$

So

$$T_\mu = \begin{pmatrix} 0 & 1-\frac{a\mu}{2} & 0 & & 1+\frac{a\mu}{2} \\ 1+\frac{a\mu}{2} & 0 & 1-\frac{a\mu}{2} & & \\ & & & & \\ 1-\frac{a\mu}{2} & & 0 & 1+\frac{a\mu}{2} & 0 \end{pmatrix}$$

Circulant matrix, Toeplitz matrix, $N \times N$.

$U^0 = [\phi_0(x_1), \phi_0(x_2) \dots \phi_0(x_i)]^T$ with $x_i = \frac{i}{N}$

b) For $\mu = \frac{1}{a} \Rightarrow 1 - a\mu = 0$

$1 + a\mu = 2$

$$T_{1/a} = \begin{pmatrix} 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & & \\ 0 & & 0 & 1 & 0 \end{pmatrix}$$

This is a cyclic permutation matrix, with $T_{1/a}^N = I$

(After N steps the wave is back to where it started).

If $T_{1/a}$ has eigenvalues λ then $T_{1/a}^N$ should have eigenvalues λ^N. But since $T_{1/a}^N = I$ then $\lambda^N = 1 \Rightarrow \lambda(T_{1/a}) = e^{2\pi i k/N} \quad k = 1..N$

All eigenvalues have unit modulus and are simple $\Rightarrow$ the method is stable.

EXTENTA #2.
2010-12-14.

1. $\dot{y} = f(y)$

$$\left(\nabla + \frac{\nabla^2}{2}\right) y_n = h\, f(y_n). \quad (1)$$

a)

$\nabla y_n = y_n - y_{n-1}$

$\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$.

$\Rightarrow (1): \; y_n - y_{n-1} + (y_n - 2y_{n-1} + y_{n-2})/2 = h f(y_n)$

$\Rightarrow y_n - y_{n-1} + \frac{y_n}{2} - y_{n-1} + \frac{y_{n-2}}{2} = h f(y_n)$.

$\Rightarrow \frac{3}{2} y_n - 2 y_{n-1} + \frac{1}{2} y_{n-2} = h f(y_n)$.

We set $y(t) = t^p$ $f(y) = pt^{p}$ and $\begin{cases} t_n = 2h \\ t_{n-1} = h \\ t_{n-2} = 0 \end{cases}$

P	y_n	y_{n-1}	y_{n-2}	$f(y_n)$	Formula
0	$(2h)^0$	$(h)^0$	$(0)^0$	$0\cdot(2h)^{-1}$	$\frac{3}{2}\cdot 1 - 2\cdot 1 + \frac{1}{2} = h\cdot 0 \Rightarrow 0 = 0$ ✓
1	$(2h)^1$	$(h)^1$	$(0)^1$	$1\cdot(2h)^0$	$\frac{3}{2}\cdot 2h - 2h = h\cdot 1 \Rightarrow h = h$ ✓
2	$(2h)^2$	$(h)^2$	$(0)^2$	$2\cdot(2h)^1$	$\frac{3}{2}\cdot 4h^2 - 2h^2 = h\cdot 4h \Rightarrow 4h^2 = 4h^2$
3	$(2h)^3$	$(h)^3$	$(0)^3$	$3\cdot(2h)^2$	$\frac{3}{2}\cdot 8h^3 - 2h^3 = h\cdot 3\cdot 8h^3 \Leftrightarrow 10h^3 = 24h$ ✗

The order of consistency is $\underline{2}$

b) $f(y_n) = 0$.

Char equ:

$\frac{3}{2} z^2 - 2z + \frac{1}{2} = 0 \Leftrightarrow z^2 - \frac{4}{3} z + \frac{1}{3} = 0$.

$\left(z - \frac{2}{3}\right)^2 - \frac{4}{9} = -\frac{1}{3} \Rightarrow \left(z - \frac{2}{3}\right)^2 = \frac{1}{9}$

$\Rightarrow z = \pm\frac{1}{3} + \frac{2}{3} \Rightarrow \begin{cases} z_1 = 1 \\ z_2 = \frac{1}{3} \end{cases}$ Inside the unitcirkle $\Rightarrow$ Zero Stable.

c) $\frac{3}{2} y_n - h f(y_n) = 2 y_{n-1} - \frac{1}{2} y_{n-2}$.

Newton's method.

2.
$$\begin{array}{c|cc} \theta & \theta & \theta \\ 1 & 1-\theta & \theta \\ \hline & 1-\theta & \theta \end{array}$$

a) $h\mathring{Y}_1 = h\lambda y_n$

$$h\mathring{Y}_2 = h\lambda(y_n + (1-\theta)h\mathring{Y}_1 + \theta h\mathring{Y}_2)$$
$$= h\lambda(1 + (1-\theta)h\mathring{Y}_1 + \theta h\mathring{Y}_2)$$
$$h\mathring{Y}_2 - h\lambda\theta h\mathring{Y}_2 = h\lambda(1+(1-\theta)h\mathring{Y}_1)$$
$$h\mathring{Y}_2(1-h\lambda\theta) = h\lambda(1+(1-\theta)h\lambda)$$
$$\Rightarrow h\mathring{Y}_2 = y_n\frac{h\lambda(1+(1-\theta)h\lambda)}{1-h\lambda\theta}$$

This gives.

$$y_{n+1} = y_n + (1-\theta)h\mathring{Y}_1 + \theta h\mathring{Y}_2 =$$
$$= y_n + y_n(1-\theta)\cdot h\lambda + y_n\theta\frac{h\lambda(1+(1-\theta)h\lambda)}{1-h\lambda\theta} =$$
$$= y_n\left(\frac{1-h\lambda\theta + (1-\theta)(1-h\lambda\theta)h\lambda + \theta h\lambda(1+(1-\theta)h\lambda)}{1-h\lambda\theta}\right) =$$
$$= y_n\left(\frac{1-h\lambda\theta + (1-h\lambda\theta-\theta+h\lambda\theta^2)h\lambda + h\lambda\theta + h\lambda\theta(1-\theta)h\lambda}{1-h\lambda\theta}\right)$$
$$y_n\left(\frac{\cancel{1-h\lambda\theta} + h\lambda - \cancel{h^2\lambda^2\theta} - \cancel{h\lambda\theta} + \cancel{h^2\lambda^2\theta^2} + \cancel{h\lambda\theta} + \cancel{h^2\lambda^2\theta} - \cancel{h^2\lambda^2\theta^2}}{1-h\lambda\theta}\right) =$$
$$= y_n\left(\frac{1-h\lambda\theta+h\lambda}{1-h\lambda\theta}\right) = y_n\left(\frac{1+(1-\theta)h\lambda}{1-h\lambda\theta}\right)$$

Thus $\Rightarrow R(\lambda h) = \dfrac{1+(1-\theta)h\lambda}{1-h\lambda\theta}$

b) To be A-stable it must be implicit or trapezoidal. This holds for $\theta = \frac{1}{2}$ (trapezoidal) or $\theta = 1$ (implicit euler)

$0 < \theta < 1$ for large $h\lambda$:

$$|R(h\lambda)| \to \left|\frac{(1-\theta)\cancel{h\lambda}}{-\cancel{h\lambda}\theta}\right| = \frac{1-\theta}{\theta} \leq 1 \text{ if } \frac{1}{2} \leq \theta \leq 1.$$

$M'' = q$; $u'' = \frac{M}{EI}$ $M(0) = M(10) = 0$

$u(0) = u(10) = 0$

a) $\frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} = \frac{f_{n-1} + 10 f_n + f_{n+1}}{12}$

$x) = \frac{q}{24EI}(1000x - 20x^3 + x^4)$

The Taylor series around x_n give:

$(x_{n-1}) = y(x_n - \Delta x) = y - \frac{y'}{1!}\Delta x + \frac{y''}{2!}\Delta x^2 - \frac{y^{(3)}}{3!}\Delta x^3 + \frac{y^{(4)}}{4!}\Delta x^4 - \frac{y^{(5)}}{5!}\Delta x^5 + O(\Delta x^6)$

$(x_{n+1}) = y(x_n + \Delta x) = y + \frac{y'}{1!}\Delta x + \frac{y''}{2!}\Delta x^2 + \frac{y^{(3)}}{3!}\Delta x^3 + \frac{y^{(4)}}{4!}\Delta x^4 + \frac{y^{(5)}}{5!}\Delta x^5 + O(\Delta x^6)$

This gives:

$y_{n-1} + y_{n+1} = 2y + 2\frac{y''}{2!}\Delta x^2 + 2\frac{y^{(4)}}{4!}\Delta x^4 + O(\Delta x^6) = 2y + y''\Delta x^2 + \frac{y^{(4)}}{12}\Delta x^4 + O(\Delta x^6)$

$= 2y_n + \Delta x^2\left(y'' + \frac{\Delta x^2}{12}y^{(4)}\right) + O(\Delta x^6)$

$\frac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2} = y''(x_n) + \frac{\Delta x^2}{12}y^{(4)} + O(\Delta x^4)$

(S)

$$\frac{f(x_{n-1}) + 10f(x_n) + f(x_{n+1})}{12} = \begin{bmatrix} y'' = f \\ y^{(4)} = f'' \end{bmatrix} =$$

$= f(x_n) + \frac{f(x_{n-1}) - 2f(x_n) + f(x_{n+1})}{12} = y''(x_n) + \frac{\Delta x^2}{12}y^{(4)} + O(\Delta x^4)$

Most $O(\Delta x^4)$ => order of consistency at least 4.

b) We sees round-off errors.

$y'' + d(x)y = \lambda y$ $y'(0) = y(1) = 0$ $d(x) \geq 0 \in [0,1]$

Neumann boundary condition at $x = 0$

So $\Delta x = \frac{1}{(N + \frac{1}{2})}$ Neumann => $\frac{y_1 - y_0}{\Delta x} = 0 \Rightarrow y_1 = y_0$

$n = 1 \ldots N$

$> \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2} + d(x_n)y_n = \lambda_{\Delta x} y_n$

$D = \text{diag}(d(x_1), d(x_2) \ldots d(x_N))$

This gives: $\underbrace{(T_{\Delta x} + D)}_{A} = \lambda_{\Delta x} y_n$

5. $t \geq 0$, $x \in [0,1]$ $u_j^n \approx u(j \cdot \Delta x, n \cdot \Delta t)$

a) $u_t = u_{xx} + f(u)$ Reaction-diffusion.

$u_t + uu_x = 0$ Inviscid Burger's equation

$u_t = u_{xx} + u_x$ Convection-diffusion.

$u_t + u_x = 0$ Advection.

$u_{tt} = u_{xx}$ Wave equation

6. $u_t + au_x = 0$ $0 \leq x \leq 1$ $t \geq 0$

$u(x,0) = \phi_0(x)$

$$u_j^{n+1} = \frac{u_{j+1}^n + u_{j-1}^n}{2} - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j+1}^n$$

a) $\mu = \frac{\Delta t}{\Delta x}$

EXTENTA #3.

Numerical Methods for Differential Equations FMNN10/NUMN12
Final exam 2011-12-12 **Grades to be announced 2011-12-21**
Solution sketch by Gustaf Söderlind

Exam duration 14:00 – 19:00. A minimum of 15 points out of 30 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(5p)** The two-step Adams-Moulton method for $y' = f(y)$; $y(0) = y_0$ can be written

$$\nabla y_n = (b_0 + b_1\nabla + b_2\nabla^2)hf(y_n),$$

where ∇ is the backward difference operator.

(a) Determine the coefficients b_j so that the method is of consistency order $p = 3$. (3p)

(b) Show that the method is zero stable. (1p)

(c) As the method is implicit, we need to solve a nonlinear equation on each step. Formulate this nonlinear equation, clearly indicating what variable to solve for. (1p)

Solution. Rewrite the formula as

$$\begin{aligned} y_n - y_{n-1} &= b_0hf_n + b_1(hf_n - hf_{n-1}) + b_2(hf_n - 2hf_{n-1} + hf_{n-2}) \\ &= (b_0 + b_1 + b_2)hf_n - (b_1 + 2b_2)hf_{n-1} + b_2hf_{n-1}, \end{aligned}$$

and insert $y = P(t)$, $f = P'(t)$ for $P(t) = t^m$ with $t_n = 2h$, $t_{n-1} = h$ and $t_{n-2} = 0$. For $P(t) \equiv 1$ and $P'(t) = 0$ we get the left hand side $1 - 1 = 0$ and the right hand side 0, so the formula holds.

For $P(t) = t$ and $P'(t) = 1$ we get the left hand side $(2h) - (h) = h$ and the right hand side

$$(b_0 + b_1 + b_2)h - (b_1 + 2b_2)h + b_2h = b_0h,$$

which requires $b_0 = 1$.

For $P(t) = t^2$ and $P'(t) = 2t$ we get the left hand side $(2h)^2 - (h^2) = 3h^2$ and the right hand side (using $b_0 = 1$)

$$\begin{aligned} &(1 + b_1 + b_2)h \cdot 2 \cdot (2h) - (b_1 + 2b_2)h \cdot 2 \cdot (h) + 0 \\ &= (4 + 4b_1 + 4b_2 - 2b_1 - 4b_2)h^2 = (4 + 2b_1)h^2, \end{aligned}$$

so we get the requirement $4 + 2b_1 = 3$, which yields $b_1 = -1/2$.

Finally, for $P(t) = t^3$ and $P'(t) = 3t^2$, we get the left hand side $(2h)^3-(h)^3 = 7h^3$ and the right hand side (using $b_0 = 1$ and $b_1 = -1/2$)

$$(\frac{1}{2} + b_2)h \cdot 3 \cdot (2h)^2 - (2b_2 - \frac{1}{2})h \cdot 3 \cdot (h)^2 = (\frac{15}{2} + 6b_2)h^2,$$

so the left hand and right hand sides are equal if $6b_2 = -1/2$, which gives $b_2 = -1/12$.

For zero stability, investigate the differential equation $y' = 0$. The method then gives

$$y_n - y_{n-1} = 0$$

with characteristic equation $z-1 = 0$. There is only one root, at $z = 1$, so the root condition is fulfilled and the method is zero stable.

Finally, the method is implicit, and one needs to solve for y_n on every step. Collecting the unknowns on the left hand side, we get the equation

$$y_n - (b_0 + b_1 + b_2)hf(y_n) = \psi,$$

where the vector ψ only depends on previously computed data. Using the coefficients above, the equation reads

$$y_n - \frac{5}{12}hf(y_n) = \psi.$$

2. **(5p)** Consider the 2-stage implicit Runge–Kutta method

$$\begin{aligned} hY_1' &= hf(t_n, y_n) \\ hY_2' &= hf(t_n + h, y_n + (hY_1' + hY_2')/2) \\ y_{n+1} &= y_n + (hY_1' + hY_2')/2) \end{aligned}$$

   (a) Write down the method's Butcher tableau. (1p)

   (b) Find its stability function $R(h\lambda)$ and express it as a rational function. (2p)

   (c) Is the method A-stable? (2p)

Solution. The Butcher tableau is

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

This is the *trapezoidal rule*. In order to find the stability function, we consider $y' = \lambda y$ and take one step, starting from $y_0 = 1$. We get

$$\begin{aligned} hY_1' &= h\lambda \\ hY_2' &= h\lambda \cdot (1 + h\lambda/2 + hY_2'/2), \end{aligned}$$

and solving for hY_2' gives

$$hY_2' = h\lambda \frac{1 + h\lambda/2}{1 - h\lambda/2}.$$

Inserting this into the updating formula gives

$$y_1 = 1 + h\lambda/2 + \frac{h\lambda}{2} \frac{1 + h\lambda/2}{1 - h\lambda/2} = \frac{1 + h\lambda/2}{1 - h\lambda/2} = R(h\lambda).$$

When we look at A-stability, we see that R has no poles in the left half-plane, and we find

$$|R(\mathrm{i}\omega)|^2 = \left|\frac{1 + i\omega/2}{1 - i\omega/2}\right|^2 = \frac{(1 + i\omega/2)(1 - i\omega/2)}{(1 - i\omega/2)(1 + i\omega/2)} \equiv 1,$$

so the method is A-stable.

3. **(5p)** Construct a second order discretization of the nonlinear two-point boundary value problem

$$\begin{aligned} &y'' + yy' - y = f(x) \\ &y(0) = \alpha, \qquad y'(1) = \beta. \end{aligned}$$

Introduce a grid and discretize with a standard second order method. Be careful to explain your notation, and *pay special attention to how the boundary conditions enter the system.*

Solution. As we have a Neumann condition at the right, we take $\Delta x = 1/(N + \frac{1}{2})$ with N internal grid points located at $x_j = j \cdot \Delta x$ for $j = 1 : N$. Noting that $(x_N + x_{N+1})/2 = 1$, we approximate the boundary condition $y'(1) = \beta$ by

$$\frac{y_{N+1} - y_N}{\Delta x} = \beta,$$

which gives $y_{N+1} = y_N + \beta\Delta x$. Discretizing the equation, we get

$$\frac{y_{j-1} - 2y_j + y_{j+1}}{\Delta x^2} - y_j \cdot \frac{y_{j-1} - y_{j+1}}{2\Delta x} - y_j = f(x_j); \qquad j = 1 : N.$$

Inserting the boundary values $y_0 = \alpha$ and $y_{N+1} = y_N + \beta\Delta x$ into the first and last equations respectively, we get

$$\begin{aligned} \frac{\alpha - 2y_1 + y_2}{\Delta x^2} - y_1 \cdot \frac{\alpha - y_2}{2\Delta x} - y_1 &= f(x_1) \\ \frac{y_{N-1} - 2y_N + y_N + \beta\Delta x}{\Delta x^2} - y_N \cdot \frac{y_{N-1} - (y_N + \beta\Delta x)}{2\Delta x} - y_N &= f(x_N). \end{aligned}$$

Simplifying these equations, moving constant terms to the right hand side and collecting all unknowns on the left hand side, the complete second order discretization becomes a nonlinear system of equations,

$$
\begin{aligned}
\frac{-2y_1 + y_2}{\Delta x^2} - y_1 \cdot \frac{\alpha - y_2}{2\Delta x} - y_1 &= f(x_1) - \frac{\alpha}{\Delta x^2} \\
\frac{y_{j-1} - 2y_j + y_{j+1}}{\Delta x^2} - y_j \cdot \frac{y_{j-1} - y_{j+1}}{2\Delta x} - y_j &= f(x_j); \qquad j = 2 : N-1 \\
\frac{y_{N-1} - y_N}{\Delta x^2} - y_N \cdot \frac{y_{N-1} - y_N - \beta\Delta x}{2\Delta x} - y_N &= f(x_N) - \frac{\beta}{\Delta x}.
\end{aligned}
$$

4. **(5p)** The Hermite differential equation is a Sturm–Liouville eigenvalue problem of the form

$$-u'' + 2xu' = \lambda u,$$

where we shall consider the case

$$u(0) = 0, \qquad u'(L) = 0.$$

Construct a second order discretization of this problem, and make sure to represent the boundary conditions to 2nd order accuracy. Write down all details, such as how you have selected your mesh size Δx, where the grid points are located, and how many they are. Finally, state the linear algebraic eigenvalue problem that results in matrix-vector form.

Solution. With a Neumann condition at the right, we take $\Delta x = L/(N + \frac{1}{2})$ with N internal grid points located at $x_j = j \cdot \Delta x$ for $j = 1 : N$. Noting that $(x_N + x_{N+1})/2 = 1$, we approximate the boundary condition $u'(L) = 0$ by

$$\frac{u_{N+1} - u_N}{\Delta x} = 0,$$

implying that $u_{N+1} = u_N$. Discretizing the equation, we get

$$-\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} - 2x_j \cdot \frac{u_{j-1} - u_{j+1}}{2\Delta x} = \lambda u_j; \qquad j = 1 : N.$$

Inserting boundary conditions, we get

$$
\begin{aligned}
-\frac{-2u_1 + u_2}{\Delta x^2} + u_2 &= \lambda u_1 \\
-\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} - j \cdot (u_{j-1} - u_{j+1}) &= \lambda u_j; \qquad j = 2 : N-1 \\
-\frac{u_{N-1} - u_N}{\Delta x^2} - N \cdot (u_{N-1} - u_N) &= \lambda u_N.
\end{aligned}
$$

In matrix-vector form, the system can be written $B_{\Delta x}u = \lambda u$, where the tridiagonal matrix $B_{\Delta x}$ is given by

$$\frac{1}{\Delta x^2}\begin{pmatrix} 2 & \Delta x^2 - 1 & 0 & 0 \\ -1 - 2\Delta x^2 & 2 & 2\Delta x^2 - 1 & 0 \\ & & \ddots & 0 \\ & -1-(N-1)\Delta x^2 & 2 & (N-1)\Delta x^2 - 1 \\ 0 & \dots & -1 - N\Delta x^2 & N\Delta x^2 + 1 \end{pmatrix}.$$

5. **(5p)** For $t \geq 0$ and $x \in [0,1]$, the following time-dependent partial differential equations are given. For each one of them, state whether they are parabolic or hyperbolic, give the structure of the CFL condition that would result if one chooses an explicit time stepping method, and state for what equations an implicit time stepping method would be preferable.

   (a) The *convection-diffusion equation* $u_t = u_x + u_{xx}$
   (b) The *wave equation* $u_{tt} = c^2 u_{xx}$
   (c) The *Schrödinger equation* $\mathrm{i}u_t = u_{xx}$
   (d) The *Korteweg–de Vries equation* $u_t + uu_x = -u_{xxx}$
   (e) The *inviscid Burgers equation* $u_t + uu_x = 0$

   Solution.
   (a) Parabolic. $\frac{\Delta t}{\Delta x^2} \leq 1$. Better to use an implicit method.
   (b) Hyperbolic. $\frac{\Delta t}{\Delta x} \leq 1$. An explicit method is satisfactory.
   (c) Hyperbolic. $\frac{\Delta t}{\Delta x^2} \leq 1$. Better to use an implicit method.
   (d) Hyperbolic. $\frac{\Delta t}{\Delta x^3} \leq 1$. Better to use an implicit method.
   (e) Hyperbolic. $\frac{\Delta t}{\Delta x} \leq 1$. An explicit method is satisfactory.

6. **(5p)** The Lax–Wendroff method for the advection equation with $a > 0$ and periodic boundary conditions,

$$\begin{aligned} &u_t + au_x = 0, \quad 0 \leq x \leq 1, \quad t \geq 0, \\ &u(x,0) = g(x), \end{aligned}$$

   is

$$u_l^{n+1} = \frac{a\mu}{2}(1 + a\mu)u_{l-1}^n + (1 - a^2\mu^2)u_l^n - \frac{a\mu}{2}(1 - a\mu)u_{l+1}^n,$$

   where $\mu = \Delta t/\Delta x$ is the Courant number.

(a) Rewrite this scheme as a matrix–vector recursion $U^{n+1} = C_\mu U^n$ with $U^n = [u_1^n, u_2^n, \ldots, u_N^n]^{\mathrm{T}}$, giving the matrix C_μ and the vector U^0, taking care to define Δx in accordance with the matrix dimension N. Is the matrix symmetric, skew-symmetric, nonsymmetric, circulant, Toeplitz (give *all characterizations that apply*), or of some other type? (3p)

(b) Put $\mu = 1/a$ in the matrix. Calculate its eigenvalues and conclude that the method is stable. (2p)

Solution. The $N \times N$ matrix C_μ is

$$C_\mu = \begin{pmatrix} 1-a^2\mu^2 & -a\mu(1-a\mu)/2 & 0 & a\mu(1+a\mu)/2 \\ a\mu(1+a\mu)/2 & 1-a^2\mu^2 & -a\mu(1-a\mu)/2 & \\ & \ddots & 1-a^2\mu^2 & \\ & & & \\ -a\mu(1-a\mu)/2 & 0 & a\mu(1+a\mu)/2 & 1-a^2\mu^2 \end{pmatrix}.$$

It is a nonsymmetric, Toeplitz, circulant matrix (due to the periodic boundary conditions). The vector $U^0 = [g(x_1), g(x_2), \ldots, g(x_N)]^{\mathrm{T}}$ represents the initial condition, with $x_j = j/N$, i.e., $\Delta x = 1/N$.

In case $a\mu = 1$ we get the matrix

$$C_{1/a} = \begin{pmatrix} 0 & 0 & & & 1 \\ 1 & 0 & & & \\ & & & \ddots & \\ 0 & & 0 & 1 & 0 \end{pmatrix}.$$

This is a cyclic permutation matrix, with $C_{1/a}^N = I$. (After N steps, the wave has come back to its original position.)

If $C_{1/a}$ has eigenvalues λ, then $C_{1/a}^N$ has eigenvalues λ^N. But since $C_{1/a}^N = I$, we have $\lambda^N = 1$, implying

$$\lambda_k[C_{1/a}] = \mathrm{e}^{2\pi \mathrm{i}k/N}; \qquad k = 1, \ldots, N.$$

Because all eigenvalues have unit modulus and are simple, the method is stable.

Extenta 2011-12-12 (1)

1 $\nabla y_n = (b_0 + b_1\nabla + b_2\nabla^2) h f(y_n)$ $y' = f(y)$ $y(0) = y_0$

Remember:
$\nabla y_n = y_n - y_{n-1}$
$\Delta y_n = y_{n+1} - y_n$

BDO $\nabla y_n = y_n - y_{n-1}$

$y_n - y_{n-1} = b_0 h f_n + b_1 h(f_n - f_{n-1}) + b_2 h(f_n - 2f_{n-1} + f_{n-2}) =$ ← (1) Expand

$= b_0 h f_n + b_1 h f_n - b_1 h f_{n-1} + b_2 h f_n - 2b_2 h f_{n-1} + b_2 h f_{n-2}$

$= h f_n (b_0 + b_1 + b_2) - h f_{n-1}(b_1 + 2b_2) + b_2 h f_{n-2}$ ← (2) Simplify

$y = P(t) \Rightarrow f = P'(t)$ $P(t) = t^m$ ← Define the polynomial.

$t_n = 2h$ $t_{n-1} = h$ $t_{n-2} = 0$ ← Define time steps.

Order $m = 0$
$P(t) = t^0 = 1$ $P'(t) = 0 \cdot t^0 = 0$ Try orders up to 3.

$\Rightarrow 1 - 1 = 0$

Order $m = 1$ $t^1 = t$ $1 \cdot t^0 = 1$
$2h - h = h(b_0 + b_1 + b_2) - h(b_1 + 2b_2) + h b_2 \Leftrightarrow$
$\Leftrightarrow h = h(b_0 + \cancel{b_1} + \cancel{b_2} - \cancel{b_1} - \cancel{2b_2} + \cancel{b_2}) = h b_0$
$\Rightarrow b_0 = 1.$

Order $m = 2$ $y = t^2$ $f = 2t$

$(2h)^2 - h^2 = h \cdot 2 \cdot 2h(1 + b_1 + b_2) - h \cdot 2 \cdot h(b_1 + 2b_2) + b_2 \cdot h \cdot 0 \Leftrightarrow$
$\Leftrightarrow 3h^2 = h^2(4 + 4b_1 + \cancel{4b_2} - 2b_1 - \cancel{4b_2}) = h^2(4 + 2b_1)$
$\Rightarrow 2b_1 = -1 \Rightarrow b_1 = -\frac{1}{2}.$

Order $m = 3$ $y = t^3$ $f = 3t^2$

$(2h)^3 - h^3 = h \cdot 3 \cdot (2h)^2 (\frac{1}{2} + b_2) - h \cdot 3 \cdot h^2 \cdot (-\frac{1}{2} + 2b_2) + b_2 \cdot h \cdot 0 =$

$= h^3(6 + 12b_2 + \frac{3}{2} - 6b_2) =$

$\Rightarrow 7 = \frac{15}{2} + 6b_2 \Rightarrow b_2 = -\frac{1}{2 \cdot 6} = -\frac{1}{12}$

$\Rightarrow \begin{cases} b_0 = 1 \\ b_1 = -\frac{1}{2} \\ b_2 = -\frac{1}{12} \end{cases}$

b) For zero-stability it must fulfill the root condition. We put $y' = 0$ and get $y_n - y_{n-1} = 0$ which leads to the characteristic equation

Root cond. Simple ~~unit~~ modulus root ≤ 1.

$z - 1 = 0$. Ther is only one root at $z = 1$ $\begin{cases} \text{* No multiple roots} \\ |z| \leq 1. \end{cases}$

It is zero stable

c) The method is implicit, so we collect the unknowns (loch new step) on the left hand side:

$y_n - (b_0 + b_1 + b_2) h f(y_n) = \phi$ where ϕ depends on prev. computed data.

Gather unknowns on left hand side!!

With the values from a) we get:

$y_n - \frac{5}{12} h f(y_n) = \phi$

$|a+bi| = \sqrt{a^2+b^2}$

2

$$hY_1' = hf(t_n, y_n)$$
$$hY_2' = hf(t_n + h, y_n + (hY_1' + hY_2')/2)$$
$$y_{n+1} = y_n + (hY_1' + hY_2')/2$$

a)
$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

b) $y' = \lambda y$ Always start with this!

$$\Rightarrow \begin{cases} hY_1' = h\lambda \\ hY_2' = h\lambda\left(1 + \frac{h\lambda}{2} + \frac{hY_2'}{2}\right) \end{cases}$$

$h\lambda = hf(t_n, y_n)$ since $\lambda y = y' = f(y_n, t_n)$

$$hY_2'\left(1 - \frac{h\lambda}{2}\right) = h\lambda\left(1 + \frac{h\lambda}{2}\right) \Rightarrow hY_2' = h\lambda \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}$$ ~~Expand~~ Solve hY_2'

For the updated step we then get

$$y_1 = 1 + \frac{h\lambda}{2} + \frac{1}{2}h\lambda \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} = 1 + \frac{h\lambda}{2} + \frac{\frac{h\lambda}{2}\left(1 + \frac{h\lambda}{2}\right)}{1 - \frac{h\lambda}{2}}$$ Insert into y_1

$$= \frac{1 \cancel{- \frac{h\lambda}{2}} \cancel{+ \frac{h\lambda}{2}} \cancel{- \left(\frac{h\lambda}{2}\right)^2} + \frac{h\lambda}{2} \cancel{+ \left(\frac{h\lambda}{2}\right)^2}}{1 - \frac{h\lambda}{2}} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} = R(h\lambda)$$

c) A-stability:

$$|R(i\omega)|^2 = \left|\frac{1 + \frac{i\omega}{2}}{1 - \frac{i\omega}{2}}\right|^2 = \frac{\left(\cancel{\sqrt{1 + (\omega/2)^2}}\right)^2}{\left(\cancel{\sqrt{1 + (\omega/2)^2}}\right)^2} = 1$$

A stability:
$|R(i\omega)| \leq 1$
Positive real parts!

$1 - \frac{i\omega}{2} = 0 \Rightarrow i\omega = 2 \Rightarrow$ {no poles in the left half plane}

In the unit circle.

$\Rightarrow$ A-stable.

$y'' + yy' - y = f(x)$

$y(0) = \alpha \quad y'(1) = \beta.$

Neumann condition @ right endpoint

$\Rightarrow \Delta x = \frac{1}{N+\frac{1}{2}}$ with N interior grid points located at $x_i = i \cdot \Delta x$ where $i = 1:N$

Special for Neumann!

Thus $\frac{x_N + x_{N+1}}{2} = 1$

So $y'(1) = \beta \approx \frac{y_{N+1} - y_N}{\Delta x} = \beta \Rightarrow y_{N+1} = y_N + \beta \Delta x$ Approximate the Neumann

We have:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + y_i \frac{y_{i+1} - y_{i-1}}{2\Delta x} - y_i = f(x_i) \qquad i = 1:N.$$

$$\frac{y_0 - 2y_1 + y_2}{\Delta x^2} + y_1 \frac{y_2 - y_0}{2\Delta x} - y_1 = f(x_1)$$

$$\frac{y_{N-1} - 2y_N + y_{N+1}}{\Delta x^2} + y_N \frac{y_{N+1} - y_{N-1}}{2\Delta x} - y_N = f(x_N).$$

With BC $y_0 = \alpha$ and $y_{N+1} = y_N + \beta \Delta x$ we get:

Discretize

$$\frac{\alpha - 2y_1 + y_2}{\Delta x^2} + y_1 \frac{y_2 - \alpha}{2\Delta x} - y_1 = f(x_1)$$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} + y_i \frac{y_{i+1} - y_{i-1}}{2\Delta x} - y_i = f(x_i) \qquad i = 1:N$$

$$\frac{y_{N-1} - y_N + \beta \Delta x}{\Delta x^2} + y_N \frac{y_N + \beta \Delta x - y_{N-1}}{2\Delta x} - y_N = f(x_N)$$

After simplification we get:

$$\frac{-2y_1 + y_2}{\Delta x^2} - y_1 \frac{\alpha - y_2}{2\Delta x} - y_1 = f(x_1) - \frac{\alpha}{\Delta x^2}$$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} - y_i \frac{y_{i-1} - y_{i+1}}{2\Delta x} - y_i = f(x_i) \qquad i = 2:N-1$$

Don't forget we added the boundary conditions!

$$\frac{y_{N-1} - y_N}{\Delta x^2} - y_N \frac{y_{N-1} - y_N - \beta \Delta x}{2\Delta x} - y_N = f(x_N) - \frac{\beta}{\Delta x}$$

Gather all unknowns on lhs.!

4. $-u'' + 2xu' = \lambda u$ $\quad u(0)=0 \quad u'(1)=0$

Note the Neumann condition at the right endpoint, thus we have

$\Delta x = \frac{L}{N+\frac{1}{2}}$ with N interior grid points where $x_j = j \cdot \Delta x$, $j = 1:N$.

Since $\frac{x_N + x_{N+1}}{2} = 1$ we approximate the Neumann condition $u'(L) = 0$

with $\frac{u_{N+1} - u_N}{\Delta x} = 0$ which gives $u_{N+1} = u_N$.

Discretizing the equation we get:

$$-\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} + 2x_j \frac{u_{j-1} - u_{j+1}}{2\Delta x} = \lambda u_j \qquad j = 1:N$$

With the boundary conditions we get:

$$-\frac{-2u_1 + u_2}{\Delta x^2} + 2x_1 \frac{-u_2}{2\Delta x} = \lambda u_1$$

$$-\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} + 2x_j \frac{u_{j-1} - u_{j+1}}{2\Delta x} = \lambda u_j \qquad j = 1:N.$$

$$-\frac{u_{N-1} - 2u_N + u_N}{\Delta x^2} + 2x_N \frac{u_{N-1} - u_N}{2\Delta x} = \lambda u_N$$

Further with $x_j = j \cdot \Delta x$ we get:

Same as prev

$$-\frac{-2u_1 + u_2}{\Delta x^2} + u_2 = \lambda u_1$$

$$-\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} + j(u_{j-1} - u_{j+1}) = \lambda u_j \qquad j = 2:N-1$$

$$-\frac{u_{N-1} - 2u_N + u_N}{\Delta x^2} + N(u_{N-1} - u_N) = \lambda u_N$$

The linear algebraic eigenvalue problem can be written as

$B_{\Delta x} u = \lambda u$ where $B_{\Delta x}$ is a tridiagonal matrix as follows:

$$B_{\Delta x} = \frac{1}{\Delta x^2} \begin{pmatrix} 2 & \Delta x^2 - 1 & 0 & 0 \\ -1\ \cancel{2\Delta x^2} & 2 & 2\Delta x^2 - 1 & 0 \\ & -1 + (N-1)\Delta x^2 & 2 & (N-1)\Delta x^2 - 1 \\ 0 & \cdot & -1 - N\Delta x^2 & 1 + N\Delta x^2 \end{pmatrix}$$

Check that the matrix works with all points!

5. a) $u_t = u_x + u_{xx}$. Except for schr. per: $u_t = \Delta x$, hyper $u_{tt} = \Delta x$!!! or $u_t = a(u) u_x$!!!

Parabolic - CFL: $\frac{\Delta t}{\Delta x^2} \leq 1$ Implicit method preferable.

$\hookrightarrow$ Check order of the derivatives.

b) $u_{tt} = c^2 u_{xx}$

Hyperbolic - CFL $= \frac{\Delta t}{\Delta x} \leq 1$ Explicit method is OK.

$\leftarrow$ same order (?)

c) $u_t = u_{xx}$

Hyperbolic - CFL: $\frac{\Delta t}{\Delta x^2} \leq 1$ Implicit method preferable.

d) $u_t + u u_x = - u_{xxx}$

Hyperbolic - CFL: $\frac{\Delta t}{\Delta x^3} \leq 1$ Implicit method preferable.

e) $u_t + u u_x$ - CFL $= \frac{\Delta t}{\Delta x} \leq 1$ Explicit method is OK.

6. $u_t + a u_x = 0$ $\quad 0 \leq x \leq 1$ $\quad t \geq 0$ $\quad a > 0$

$u(x,0) = g(x)$

$$\Rightarrow \quad u_\ell^{n+1} = \frac{a\mu}{2}(1 + a\mu) u_{\ell-1}^n + (1 - a^2\mu^2) u_\ell^n - \frac{a\mu}{2}(1 - a\mu) u_{\ell+1}^n \qquad \left(\mu = \frac{\Delta t}{\Delta x}\right)$$

a) $U^{N+1} = C_\mu U^N$

$(N \times N)$ matrix C_μ is

$$C_\mu = \begin{pmatrix} 1 - a^2\mu^2 & -\frac{a\mu}{2}(1 - a\mu) & 0 & \frac{a\mu}{2}(1 + a\mu) \\ \frac{a\mu}{2}(1 + a\mu) & 1 - a^2\mu^2 & -\frac{a\mu}{2}(1 - a\mu) & \ddots \\ & \ddots & 1 - a^2\mu^2 & \ddots \\ -\frac{a\mu}{2}(1 - a\mu) & 0 & \frac{a\mu}{2}(1 + a\mu) & 1 - a^2\mu^2 \end{pmatrix}$$

Nonsymmetric Toeplitz Circular

$U^0 = [g(x_1), g(x_2) \ldots g(x_N)]^T$ is the initial condition

with $x_i = \frac{i}{N} \Rightarrow \Delta x = \frac{1}{N}$.

b) $\mu = \frac{1}{a}$. We get:

$$1 - a^2 \cdot \frac{1}{a^2} = 1 - 1 = 0$$

$$-\frac{a}{2} \cdot \frac{1}{a}\left(1 - a \cdot \frac{1}{a}\right) = 0 \quad \Rightarrow C_{1/a} = \begin{pmatrix} 0 & 0 & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\frac{a}{2} \cdot \frac{1}{a}\left(1 + a \cdot \frac{1}{a}\right) = \frac{1}{2} \cdot 2 = 1.$$

So if $C_{1/a}$ have eigenvalues λ then $C_{1/a}^N$ has eigenvalues λ^N

However $C_{1/a}^N = I$, since after N steps the wave has come back to its original position, so we have $\lambda^N = 1 \Rightarrow \lambda_k[C_{1/a}] = e^{2\pi i k/N}$ $\quad k = 1:N$.

The method is stable because all eigenvalues have unit modulus and are simple.

EXTENTA #3.
2011-12-12

1. $y' = f(y)$; $y(0) = y_0$

$$\nabla y_n = (b_0 + b_1 \nabla + b_2 \nabla^2) h f(y_n)$$

a) $\nabla = y_n - y_{n-1}$

$\nabla^2 = y_n - 2y_{n-1} + y_{n-2}$

$\Rightarrow y_n - y_{n-1} = (b_0 + b_1 \nabla h f(y_n) + b_2 \nabla^2 h f(y_n))$

$= (b_0 f_n + b_1 h (f_n - f_{n-1}) + b_2 h (f_n - 2f_{n-1} + f_{n-2})) =$

$= ((b_0 + b_1 + b_2) h f_n - (b_1 + 2b_2) h f_{n-1} + b_2 h f_{n-2}$

$y(t) = t^p$ $f(y) = p t^{p-1}$

$t_n = 2h$

$t_{n-1} = h$

$t_{n-2} = 0$

Thives gives for p

p	$(2h)^p$ y_n	$y_n - y_{n-1}$	hf_n	f_{n-1}	f_{n-2}
0	$(2h)^0 = 1$	h^0	$0 \cdot (2h)^{-1}$	$0 \cdot (h)^{-1}$	$0 \cdot (0)^{-1}$
1	$(2h)^1$	h^1	$1 \cdot (2h)^0$	$1 (h)^0$	$1 \cdot (0)^0 = 1$
2	$4h^2$	h^2	$4h$	$2h$	0
3	$8h^3$	h^3	$12h^2$	$3h^2$	0

Thus for p=0:

$1 - 1 = 0$ ✓.

$p = 1$: $2h - h = (b_0 + b_1 + b_2) h \cdot 1 - (b_1 + 2b_2) \cdot h \cdot 1$

$\Leftrightarrow h = (b_0 - b_2) h + b_2 h \Rightarrow h = b_0 h \Rightarrow b_0 = 1$

$p = 2$: $3h^2 = (b_0 + b_1 + b_2) 4h^2 - (b_1 + 2b_2) 2h^2$

$\Leftrightarrow 3h^2 = (4b_0 + 2b_1) h^2 \Rightarrow b_1 = -\frac{1}{2}$

$p = 3$: $7h^3 = (b_0 + b_1 + b_2) 12h^3 - (b_1 + 2b_2) 3h^3$

$\Leftrightarrow 7h^3 = (12b_0 + 9b_1 + 6b_2) h^3$

$\Rightarrow 7h^3 - 12h^3 + \frac{9}{2} h^3 = 6 b_2 h^3$

$h^3 \left(-\frac{1}{2}\right) = 6 b_2 h^3 \Rightarrow b_2 = -\frac{1}{12}$

$\Rightarrow$ $b_0 = 1$, $b_1 = -\frac{1}{2}$, $b_2 = -\frac{1}{12}$.

b) $f(y_n) = 0$.
$\Rightarrow y_n - y_{n-1} = 0$.

$z - 1 = 0$ Only 1 root, $z = 1$.
On the unit circle so the method is zerostable.

c) $y_n - hf(y_n)(b_0 + b_1 + b_2) = y_{n-1}$

This gives $y_n + \frac{5}{12} hf(y_n) = y_{n-1}$

2. $h\dot{Y}_1 = hf(t_n, y_n)$
$h\dot{Y}_2 = hf(t_n + h, y_n + (h\dot{Y}_1 + h\dot{Y}_2)/2)$
$y_{n+1} = y_n + (h\dot{Y}_1 + h\dot{Y}_2)/2)$

a)
$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

b) $y' = \lambda y$.

$h\dot{Y}_1 = h\lambda$
$h\dot{Y}_2 = h\lambda(1 + \frac{h\lambda}{2} + h\dot{Y}_2/2)$
$h\dot{Y}_2 - \frac{h\lambda}{2} \cdot h\dot{Y}_2 = h\lambda(1 + \frac{h\lambda}{2})$.
$h\dot{Y}_2(1 - \frac{h\lambda}{2}) = h\lambda(1 + \frac{h\lambda}{2})$.
$\Rightarrow h\dot{Y}_2 = h\lambda \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}$

$\Rightarrow y_1 = 1 + \frac{h\lambda}{2} + \frac{h\lambda}{2} \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} =$
$= 1 + \frac{h\lambda}{2}\left(1 + \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}\right) \Rightarrow$
$\Rightarrow 1 + \frac{h\lambda}{\cancel{2}}\left(\frac{\cancel{2}}{1 - \frac{h\lambda}{2}}\right) = 1 + \frac{h\lambda}{1 - \frac{h\lambda}{2}} =$
$= \frac{1 - \frac{h\lambda}{2} + h\lambda}{1 - \frac{h\lambda}{2}} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} = R(\lambda h)$.

① A-stability = no poles in left half-plane.

$$|R(i\omega)|^2 = \left|\frac{1+\frac{i\omega}{2}}{1-\frac{i\omega}{2}}\right|^2 = \frac{(1+i\omega/2)(1-i\omega/2)}{(1-i\omega/2)(1+i\omega/2)}$$

③ $y'' + y\,y' - y = f(x)$

$y(0) = \alpha \quad y'(1) = \beta.$

Neumann condition at the right endpoint gives

$$\Delta x = \frac{1}{(N+\frac{1}{2})} \quad \text{with } N \text{ gridpoints at } x_j = j\cdot\Delta x,\ j = 1 \ldots N$$

Neumann condition is approximated by

$$\frac{y_{N+1} - y_N}{\Delta x} = \beta. \Rightarrow y_{N+1} = y_N + \beta\Delta x$$

We discretize the equation

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{\Delta x^2} + y_j \cdot \frac{y_{j+1} - y_{j-1}}{2\Delta x} - y_j = f(x_j) \quad j = 1, \ldots, N.$$

$$y_0 = \alpha;\ y_{N+1} = y_N + \beta\Delta x$$

BC

$$\frac{y_2 - 2y_1 + \alpha}{\Delta x^2} + y_1 \cdot \frac{y_2 - \alpha}{2\Delta x} - y_1 = f(x_1)$$

$$\frac{y_N + \beta\Delta x - 2y_N + y_{N-1}}{\Delta x^2} + y_N \cdot \frac{y_N + \beta\Delta x - y_{N-1}}{2\Delta x} = f(x_N)$$

$$\Rightarrow \begin{cases} \dfrac{y_2 - 2y_1}{\Delta x^2} + y_1 \cdot \dfrac{y_2 - \alpha}{2\Delta x} - y_1 = f(x_1) - \dfrac{\alpha}{\Delta x^2} \\[2ex] \dfrac{y_{j+1} - 2y_j + y_{j-1}}{\Delta x^2} + y_j \dfrac{y_{j+1} - y_{j-1}}{2\Delta x} - y_j = f(x_j) \quad j = 1 \ldots N \\[2ex] \dfrac{y_{N-1} - y_N}{\Delta x^2} + y_N \dfrac{y_N + \beta\Delta x - y_{N-1}}{2\Delta x} = f(x_N) - \dfrac{\beta}{\Delta x} \end{cases}$$

a) Parabolic $\frac{\Delta t}{\Delta x^2} \leq 1$ Implicit

b) Hyperbolic $\frac{\Delta t}{\Delta x} \leq 1$ Explicit

c) Hyperbolic $\frac{\Delta t}{\Delta x^2} \leq 1$ Implicit

d) Hyperbolic $\frac{\Delta t}{\Delta x^3}$ Implicit

e) Hyperbolic $\frac{\Delta t}{\Delta x} \leq 1$ Explicit

Numerical Methods for Differential Equations FMN130.2–081218
Gustaf Söderlind

The exam starts at 14:00 and ends at 19:00. A minimum of 15 points out of the total 30 are required to pass. These points will be added to your total project score. For final grade requirements, see home page.

You are not allowed a computer, pocket calculator, textbook, lecture notes or any other electronic or written material during the exam.

1. **(5p)** Consider the explicit midpoint method

$$y_{n+1} - y_{n-1} = 2hf(y_n).$$

   (a) Determine whether the method is zero-stable or not. (1p)

   (b) Determine the consistency order of the method. (2p)

   (c) If the method is applied to the advection equation, then stability can be determined by studying the linear test equation $\dot{y} = \mathrm{i}\omega y$ (note that in this special case we take $\lambda = \mathrm{i}\omega$). Find the condition on $h\omega$ for stability. (2p)

2. **(6p)** Consider Heun's method, given by the Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

   (a) Find its stability polynomial $P(h\lambda)$. The boundary of the stability region is given by the condition $|P(z)| = 1$. Determine its two intersections with the real axis in the complex plane. (2p)

   (b) If the method is applied to a linear system of differential equations, $\dot{y} = T_{\Delta x} y$, one gets a recursion $y_{n+1} = Q(hT_{\Delta x})y_n$, where Q is a polynomial. Find the polynomial Q. (1p)

   (c) Let $\lambda[T_{\Delta x}]$ denote the eigenvalues of $T_{\Delta x}$. What are the eigenvalues of $Q(hT_{\Delta x})$? (1p)

   (d) As the method is explicit it cannot be A-stable. A student who wants to solve the heat equation $u_t = u_{xx}$ with an explicit method knows that there will be a CFL condition, but he hopes that using Heun's method will enable him to use larger time steps than Euler's method allows with the CFL condition $\Delta t/\Delta x^2 \leq 1/2$. Is he right? (Assume that $T_{\Delta x}$ is the usual Toeplitz matrix). (2p)

3. **(4p)** The following nonlinear two-point boundary value problem is given:

$$y'' + yy' - y = g(x)$$
$$y(0) = 0, \qquad y'(1) = 0.$$

Introduce a grid and discretize with a standard second order finite difference method. Be careful to define Δx, write down all equations, and show exactly how the boundary conditions affect the system by writing down the first and the last equations separately. (4p)

4. **(4p)** In the course we have studied Sturm–Liouville eigenvalue problems, such as in the Euler buckling of a beam of unit length,

$$u'' = \lambda u$$

with various boundary conditions depending on which buckling case is considered. Here we shall consider Euler's second buckling case, for which the boundary conditions are

$$u(0) = 0, \qquad u(1) = 0.$$

If we solve this with the standard 2nd order finite difference method, as we have done in the course, we need to solve the algebraic eigenvalue problem

$$T_{\Delta x} y = \lambda y$$

with $T_{\Delta x} = \text{tridiag}(1 \quad -2 \quad 1)/\Delta x^2$. Let us call this Method A.

If instead we would work with the finite element method with piecewise linear basis functions, one needs to solve a "generalized" eigenvalue problem,

$$T_{\Delta x} y = \lambda B y$$

with $B = \text{tridiag}(1 \quad 4 \quad 1)/6$. This is method B.

Finally, for the problem $y'' = f$ there is a special finite difference method known as Cowell's method. In matrix-vector form it takes the shape $T_{\Delta x} y = Df$, with $D = \text{tridiag}(1 \quad 10 \quad 1)/12$. If this method, called C, is used to solve the eigenvalue problem, one also has to solve a "generalized" eigenvalue problem,

$$T_{\Delta x} y = \lambda D y.$$

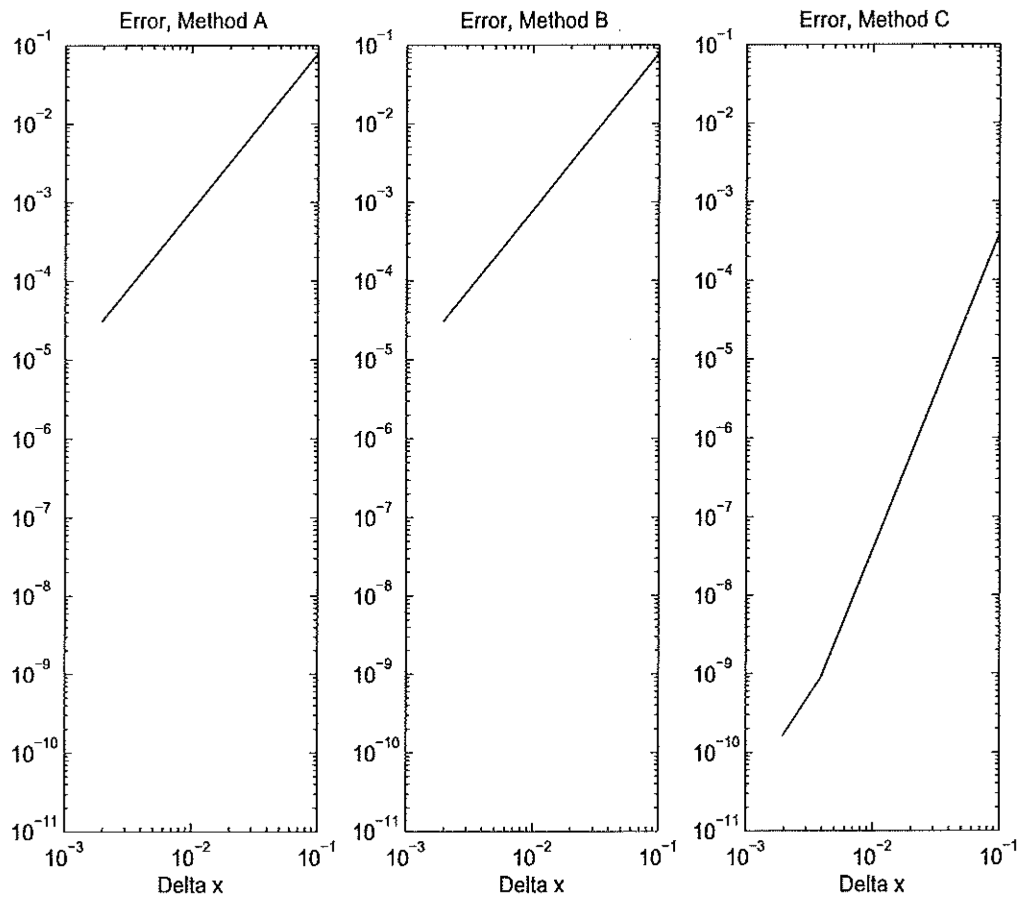


Figure 1: *Eigenvalue error* is plotted vs. step size Δx for Methods A, B and C, which use different FDM and FEM methods

The three methods were implemented and tested, with results displayed in Figure 1.

(a) Using the experimental data, determine the order of convergence for the Finite Element Method and Cowell's Method. (2p)

(b) Given that Cowell's method approximates $y'' = f(x)$ by $T_{\Delta x} y = Df$, determine its consistency order theoretically. (Hint: Because the method is symmetric its order will be even, so you do not need to bother about checking whether the method is order 1, 3, 5, and so on.)

5. **(5p)** Consider the following PDEs for $t \geq 0$ and $x, y \in [0, 1]$:

   (a) $u_{xx} + u_{yy} = f(x, y)$

   (b) $u_t + a \cdot u_x = 0$

   (c) $u_t = u_x + \frac{1}{\mathrm{Pe}} u_{xx}$

   (d) $u_t = d \cdot u_{xx} + f(u)$

   (e) $u_t + \frac{1}{2}(u^2)_x = u_{xx}$

   For each equation, classify the problem as *elliptic, parabolic or hyperbolic.* In addition, give the *name* of each equation, or, in case it has no name, name it based on the terms that enter the equation.

6. **(6p)** We are going to solve the equation

$$u_t + a u_x = 0, \quad 0 \leq x \leq 1, \quad t \geq 0,$$

   for $a > 0$ with initial condition $u(0, x) = g(x)$ and boundary condition $u(t, 0) = f(t)$, using the Euler upwind scheme

$$u_j^{n+1} = u_j^n - a\mu(u_j^n - u_{j-1}^n),$$

   where the Courant number is given by $\mu = \Delta t / \Delta x$.

   (a) Write this as a matrix–vector recursion $U^{n+1} = T_\mu U^n + F^n$ with $U^n = [u_1^n, u_2^n, \ldots, u_J^n]^{\mathrm{T}}$, taking special care to define Δx in terms of the number of grid points used (that is, J; for clarity do not hesitate to draw your grid). Give the matrix T_μ and the vector F^n, likewise defining their dimensions. (3p)

   (b) Draw the computational stencil ("beräkningsmolekyl") for this method.(1p)

   (c) Determine the eigenvalues of T_μ for $\mu = 1/a$. Is the method stable then? (2p)

LYCKA TILL — GOOD LUCK! G.S.

GOD JUL
MERRY CHRISTMAS
JOYEUX NOËL
FRÖHLICHE WEIHNACHTEN
FELIZ NAVIDAD
BUON NATALE
BON NADAL

Numerical Methods for Differential Equations FMN130 080821
Gustaf Söderlind

The exam starts at 14:00 and ends at 19:00. A minimum of 15 points out of the total 30 are required to pass. These points will be added to those you obtained in your three projects.

You are not allowed a computer, pocket calculator, textbook, lecture notes or any other electronic or written material during the exam.

1. **(5p)** Consider explicit multistep methods of the form

$$\alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h\beta_1 f(y_{n+1}) + h\beta_0 f(y_n)$$

   (a) First, specialize to methods with β_0. Determine the remaining coefficients of the method of the highest possible order, and verify the order. Note: the method must be zero-stable. (3p)

   (b) What is the highest possible order? (1p)

   (c) If you can also choose $\beta_0 \neq 0$, can you obtain a zero-stable method of a higher order than in the previous case? You may answer either by referring to a suitable theorem or by checking whether it is possible. (1p)

2. **(5p)** Consider the 2-stage Runge-Kutta method with Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

   (a) Write down the *formulas* for using this method. (2p)

   (b) Find its *stability function* $R(h\lambda)$. (2p)

   (c) Is the method *A-stable*? (1p)

3. **(5p)** Consider the following linear two-point boundary value problem:

$$y'' + 2\pi^2 y' - y = g(x)$$
$$y(0) = 0, \qquad y(1) = 1.$$

   (a) Introduce a grid and discretize with a standard second order method. Be careful to show exactly how the boundary conditions affect the system. (2p)

   (b) Use the Euclidean logarithmic norm and its properties to show that this problem has a unique solution for every right-hand side $g(x)$. (You may state and use "well-known" values for the logarithmic norms of the Toeplitz matrices involved.) (3p)

4. **(5p)** In the course we have studied Sturm–Liouville eigenvalue problems, such as in the Euler buckling of a beam of length L,

$$u'' = \lambda u$$

with various boundary conditions depending on which buckling case is considered. Here we shall consider Euler's first buckling case, for which the boundary conditions are

$$u'(0) = 0, \qquad u(L) = 0.$$

Construct a second order discretization of this problem, and make sure to represent the boundary conditions to 2nd order accuracy. Write down all details, such as how you have selected your grid size Δx, where the grid points are located, and how many they are. Finally, state the algebraic eigenvalue problem that results, by giving all matrix elements of the matrix whose eigenvalues and eigenvectors are to be determined.

5. **(5p)** Consider the following PDEs for $t \geq 0$ and $x \in [0, 1]$:

   (a) $u_t + a \cdot u_x = d \cdot u_{xx}$
   (b) $u_t + a \cdot u_x = 0$
   (c) $u_t = d \cdot u_{xx} + f(u)$
   (d) $u_t + \frac{1}{2}(u^2)_x = 0$
   (e) $u_{tt} = c^2 \cdot u_{xx}$

   For each equation, classify the problem as *elliptic, parabolic or hyperbolic.* In addition, give the *"name"* of each equation, based on the terms that enter the equation.

6. **(5p)** The Lax–Friedrichs scheme for the equation

$$u_t + au_x = 0, \quad 0 \leq x \leq 1, \quad t \geq 0,$$

with $a > 0$ and *periodic boundary conditions*, is

$$u_j^{n+1} = (u_{j+1}^n + u_{j-1}^n)/2 - a\Delta t(u_{j+1}^n - u_{j-1}^n)/(2\Delta x).$$

   (a) Let $\mu = \Delta t/\Delta x$. Construct the matrix T_μ for the system $U^{n+1} = T_\mu U^n$ with $U^n = [u_1^n, u_2^n, \ldots, u_L^n]^{\mathrm{T}}$. Is the matrix symmetric, skew-symmetric, unsymmetric, circulant, or of some other type? (2p)
   (b) Draw the computational stencil ("beräkningsmolekyl") for this method with $\mu = 1/a$. Is the method an upwind or a downwind method for this value of μ? (1p)
   (c) Determine the eigenvalues of T_μ for $\mu = 1/a$. (2p)