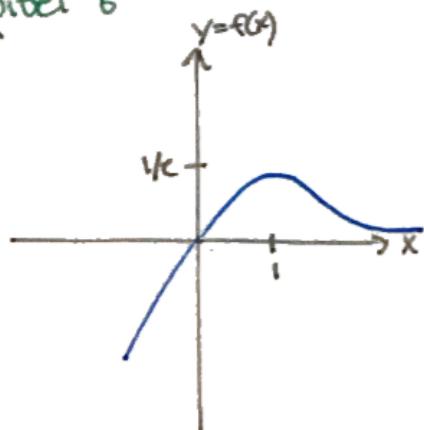


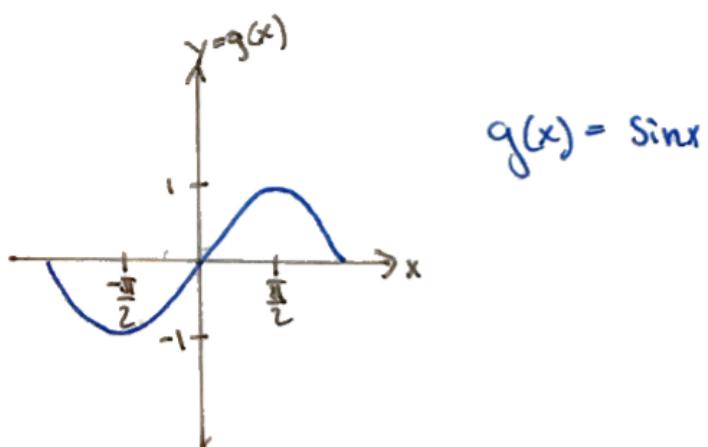
## Kapitel 6

b.1 a)



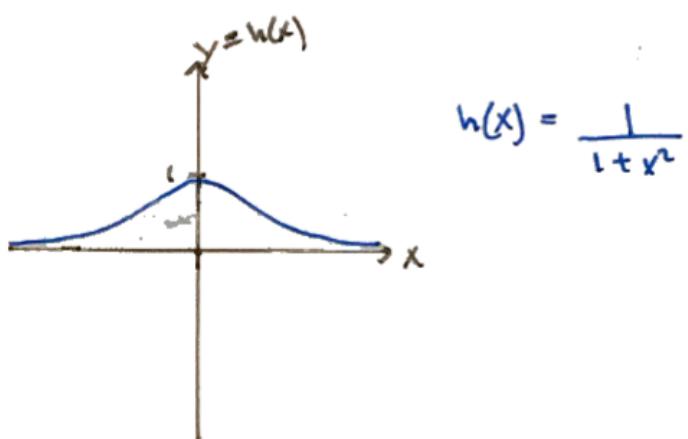
$$f(x) = xe^{-x} \Rightarrow f'(x) = e^{-x}(1-x)$$

b)



$$g(x) = \sin x$$

c)



$$h(x) = \frac{1}{1+x^2}$$

$$6.2 \quad \alpha > 0, \quad S_\alpha f(x) = f(\alpha x)$$

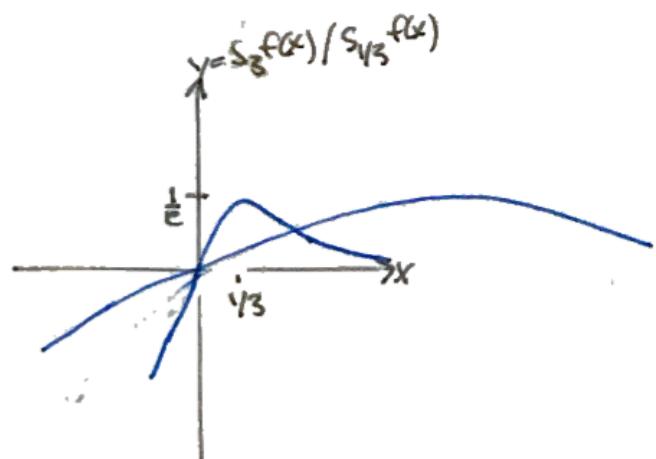
$$S_3 f(x) = f(3x) = 3x e^{-3x}$$

$$S_3 f'(x) = 3e^{-3x} (1 - 3x)$$

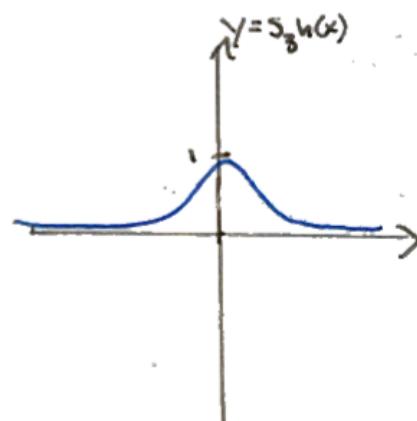
$$S_{1/3} f(x) = f(x/3) = \frac{x}{3} e^{-x/3}$$

$$S_{1/3} f'(x) = \frac{-x^2/3}{3} (1 - \frac{x}{3})$$

$$S_3 g(x) = g(3x) = \sin(3x)$$



$$S_3 h(x) = h(3x) = \frac{1}{1 + 9x^2}$$



$$6.3 \quad a) \quad \left\| x e^{-x} \right\| = \sup_{x \geq 0} |x e^{-x}| = 1/e$$

$$b) \quad \left\| \sin x \right\| = \sup_{x \in \mathbb{R}} |\sin x| = 1$$

$$c) \quad \left\| \frac{1}{1+x^2} \right\| = \sup_{x \in \mathbb{R}} \left| \frac{1}{1+x^2} \right| = 1$$

$$6.5 \text{ a) } f(x) = xe^{-x}, x \geq 0$$

$$S_n f(x) = f(nx) = nx e^{-nx} \rightarrow 0 \text{ di } n \rightarrow \infty \text{ for } x \geq 0$$

$$S_n f'(x) = ne^{-nx}(1-nx), \|S_n f(x) - f(x)\| = \|S_n f(x) - 0\| =$$

$$\|S_n f(x)\| = \sup_{x \geq 0} |nx e^{-nx}| = e^{-1} = 1/e \neq 0, \text{ punktvis uten tube}$$

likformig

$$\text{b) } g(x) = \sin x, x \in \mathbb{R}, S_n g(x) = g(nx) = \sin(nx) \text{ konvergerer  
varken punktvis eller likformig}$$

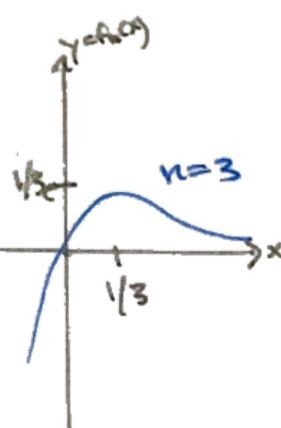
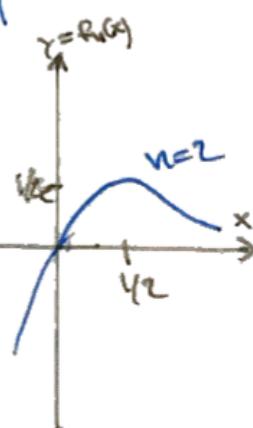
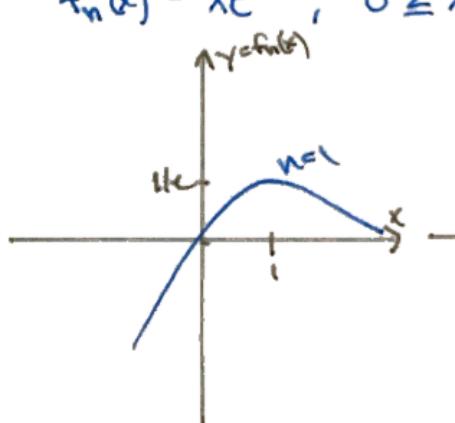
$$\text{c) } h(x) = \frac{1}{1+x^2}, x \in \mathbb{R}, S_n h(x) = h(nx) = \frac{1}{1+(nx)^2} \rightarrow 0 \text{ di } n \rightarrow \infty$$

$$\|S_n h(x) - 0\| = \sup_{x \in \mathbb{R}} \left| \frac{1}{1+(nx)^2} \right| = 1 \neq 0, \text{ punktvis uten tube}$$

$$\text{likformig, } h(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x=0 \end{cases}$$

$$\text{b.b) } f_n(x) = xe^{-nx}, 0 \leq x \leq 1$$

a)



b) Taft fölgen  $f_n(p)$  är konvergent för varje  $0 < p \leq 1$

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x e^{-nx} = 0 \quad \text{för } 0 < x \leq 1 \quad \underline{0}$$

$$x=0 \Rightarrow f(x) = 0 \cdot e^{-n \cdot 0} = 0, \text{ inkl. d. } n \rightarrow \infty \Rightarrow$$

$$f(x) = 0, \quad 0 \leq x \leq 1$$

c)

$$\|f - f_n\| = \sup_{0 \leq x \leq 1} |0 - x e^{-nx}| = \frac{1}{ne} \rightarrow 0 \text{ d. } n \rightarrow \infty$$

Ja, funktionen konvergerar likformigt mot 0 på  $[0,1]$

b.7  $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$

$$x=0 \Rightarrow f_n(x) = 0 \Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 0$$

$$(|x| < 1, x \neq 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{x^{2n}}{1+x^{2n}} = 0)$$

$$x=\pm 1 \Rightarrow f_n(x) = 1/2 \Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 1/2$$

$$|x| > 1 \Rightarrow f_n(x) = \frac{x^{2n}}{x^{2n}} \cdot \frac{1}{\frac{1}{x^{2n}} + 1} \Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 1$$

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1/2, & x = \pm 1 \\ 1, & |x| > 1 \end{cases}$$

Ej likformig konvergens på hela  $\mathbb{R}$

$$\|f - f_n\|_{[2, \infty]} = \sup_{[2, \infty]} \left| 1 - \frac{x^n}{1+x^n} \right| = \frac{1}{1+4^n} \rightarrow 0 \text{ di } n \rightarrow \infty$$

Ta. funktionen konvergerar likformigt mot l p:  $[2, \infty)$

b.8  $f(z) = \sum_{k=1}^{\infty} \frac{e^{-kz}}{k \cdot z^k}$

a) Om  $\operatorname{Re} z > 0$ ,  $z = x + iy$

$$\left\| \frac{e^{-kz}}{k \cdot z^k} \right\| = \left\| \frac{e^{-k(x+iy)}}{k \cdot z^k} \right\| = \frac{1}{k z^k} \sup_{x>0} |e^{-kx}| \leq \frac{1}{k z^k} \rightarrow 0 \text{ di } n \rightarrow \infty$$

Weierstrass M-test visar att serien konvergerar likformigt

b) Om  $f_n$  är en följd av holomorfa funktioner som konvergerar likformigt mot f är f också holomorf  $\Omega$   
 $f'_n$  konv. lokalt likformigt mot  $f'$

$$f_k(z) = \frac{e^{-kz}}{k \cdot z^k} = \frac{1}{k \cdot (ze^{-z})^k}$$

$$f'_k(z) = \frac{-ke^{-kz}}{k \cdot z^k} = -\frac{e^{-kz}}{z^k} = -\left(\frac{e^{-z}}{z}\right)^k$$

$$f'(z) = -\sum_{k=1}^{\infty} \left(\frac{e^{-z}}{z}\right)^k = -\frac{\frac{e^{-z}}{z}}{1 - \frac{e^{-z}}{z}} = \frac{\frac{e^{-z}}{z}}{\frac{e^{-z}-z}{z}} = \frac{e^{-z}}{e^{-z}-z}$$

$$c) f(z) = \int f'(z) dz = \int \frac{e^z}{e^z - 2} dz = \left[ u = e^z \atop du = e^z dz \right] =$$

$$\int \frac{du}{2-u} = -\log(2-u) + C = -\log(2-e^{-z}) + C$$

$\log$  är principal grenen (exempelvis)

$$f(0) = \sum_{k=1}^{\infty} \frac{1}{k \cdot z^k} =$$

d) Så länge  $2-e^z > 0$  är principalgrenen ok,  
 dvs hela  $\mathbb{C}$  förutom  $e^{-z} > 2 \Leftrightarrow -z > \log(2) \Leftrightarrow$   
 $z < \ln 2 + 2\pi i k, k \in \mathbb{Z}$

$$x \in [-1/2, 1/2], [-1, 1], \mathbb{R}$$

b.9 a)  $\sum_{k=0}^{\infty} k^2 x^k$

Gränsfunktioner

1.  $x \in [-1/2, 1/2]$ ,  $f(x) = 0$ ,  $|x| \leq 1/2$

$$\left\| k^2 x^k - 0 \right\| = \sup |k^2 x^k| = \frac{k^2}{2^k} \rightarrow 0 \text{ di } k \rightarrow \infty, \text{ likformigt konv.}$$

2.  $x \in [-1, 1]$

För  $x=1 \Rightarrow k^2 1^k \rightarrow \infty$ , ej punktvis konv

3.  $x \in \mathbb{R}$ , samma argument som ovan

b)  $\sum_{k=0}^{\infty} \frac{\sin kx}{1+k^2}$

$$\left\| \frac{\sin kx}{1+k^2} \right\| = \sup_{x \in \mathbb{R}} \left| \frac{\sin kx}{1+k^2} \right| = \frac{1}{1+k^2} \leq \frac{1}{k^2} \rightarrow 0 \text{ di } k \rightarrow \infty \text{ för}$$

alla intervall

b.10  $s(x) = \sum_{k=1}^{\infty} \underbrace{\frac{x^k}{(1+x^2)^k}}_{f_n(x)}$

Reella:

$$x=0 \Rightarrow f_n(x) = 0$$

$$x \neq 0 \Rightarrow f_n(x) \rightarrow 0 \text{ di } k \rightarrow \infty, f(x) = \begin{cases} 0, & x=0 \\ 0, & x \neq 0 \end{cases}, \text{ dvs alla nella}$$

ej likformigt på  $\mathbb{R}$

Komplexa:

$$z=0 \Rightarrow f_n(z) = 0 \quad \text{Sant om } |1+z^2| > 1$$

$$b.11 \quad f(x) = \sum_{k=0}^{\infty} \underbrace{\frac{1}{1+(k+x)^2}}_{g_n(x)}$$

Om  $f(x)$  konv. likformigt  $\Leftrightarrow g_n(x)$  är en följd av konvinkonvergenter funktioner.

$$|g_n(x)| = \left| \frac{1}{1+(k+x)^2} \right| = \frac{1}{1+(k+x)^2} \leq \frac{1}{k^2} \quad \text{för } x \geq 0$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$  konvergerar  $\Rightarrow f(x)$  konv enligt Weierstrass M-test

$$\int_0^1 f(x) dx = \sum_{k=0}^{\infty} \int_0^1 \frac{dx}{1+(k+x)^2} = \sum_{k=0}^{\infty} [\arctan(k+x)]_0^1 =$$

$$\sum_{k=0}^{\infty} \arctan(k+1) - \arctan(k) = \arctan(N+1) - \arctan(0) = \frac{\pi}{2}$$

$$b.13 \text{ a)} \quad f(x) = \sum_{k=0}^{\infty} \underbrace{\frac{e^{-kx}}{k^2+1}}_{g_n(x)}, \quad g(x) = 0, \quad x \in [1, 5]$$

$$\|g_n(x) - 0\| = \sup_{x \in [1, 5]} \left| \frac{e^{-kx}}{k^2+1} \right| = \frac{e^{-k}}{k^2+1} \rightarrow 0 \quad \text{di } k \rightarrow \infty \Rightarrow \text{likformig konvergens}$$

b) Fortsatende likformig konvergens for  $x \in [r, \infty)$

$$g_n'(x) = \frac{-ke^{-kx}}{k^2+1}, \quad g'(x) = 0 \quad \text{for } x \in [r, \infty)$$

$$\left\| \frac{-ke^{-kx}}{k^2+1} - 0 \right\| = \sup_{x \in [r, \infty)} \left| \frac{-ke^{-kx}}{k^2+1} \right| = e^{-kr} \cdot \frac{k}{k^2+1} \rightarrow 0 \quad \text{di } k \rightarrow \infty, r > 0$$

$$\text{Då gäller att } f'(x) = \sum_{k=0}^{\infty} \frac{-ke^{-kx}}{k^2+1}$$

c)  $y = f(x)$

$$y'' + y = \frac{e^x}{e^x - 1}$$

$$g_n''(x) = \frac{k^2 e^{-kx}}{k^2 + 1}, \quad g''(x) = 0 \quad \text{for } x \in [r, \infty), r > 0$$

$$\left\| \frac{k^2 e^{-kx}}{k^2 + 1} \right\| = e^{-kr} \cdot \frac{k}{k^2 + 1} \rightarrow 0 \quad \text{di } k \rightarrow \infty, r > 0$$

$$f''(x) = \sum_{k=0}^{\infty} \frac{k^2 e^{-kx}}{k^2 + 1}$$

$$y'' + y = \sum_{k=0}^{\infty} \left( \frac{k^2 e^{-kx}}{k^2 + 1} + \frac{e^{-kx}}{k^2 + 1} \right) = \sum_{k=0}^{\infty} e^{-kx} \left( \frac{k^2}{k^2 + 1} + \frac{1}{k^2 + 1} \right) =$$

$$\sum_{k=0}^{\infty} e^{-kx} = \frac{1}{e^x - 1} = \frac{e^x}{e^x - 1}$$

6.4  $a > 0$ ,  $f$  def. på hela  $\mathbb{R}$

$$S_a f(x) = f(ax)$$

Om  $f$  def för hela  $\mathbb{R}$   $\|S_a f\| = \|f\|$

Integ. skillnad för  $x \geq 0$  & bry  $a > 0$ , men skillnad  
om  $x \in [a, 1]$ , ex.  $f(x) = 2 - (x-4)^2$

$$\|f\| = \sup_{x \in [a, 1]} |2 - (x-4)^2| = 7$$

$$\|S_a f\| = \sup_{x \geq 0} |2 - (x-4)^2| = 2, \text{ bry } a > 0 \text{ sprider ut intervallet}$$

6.14 a)  $P(z) = \sum_{n=0}^{\infty} z^n + \sum_{n=1}^{\infty} \bar{z}^n$

Om  $|z| < 1$  är  $|\bar{z}| < 1$  & då är  $P(z)$  summan av två geometriska serier, dvs

$$P(z) = \frac{1}{z-1} + \frac{\bar{z}}{\bar{z}-1} = \frac{\bar{z}-1 + |\bar{z}|^2 - \bar{z}}{(\bar{z}-1)(\bar{z}-1)} = \frac{1 - |\bar{z}|^2}{(1-z)(1-\bar{z})} = \frac{1 - |z|^2}{1 - z\bar{z}}$$

$$b) \quad 0 < r < 1$$

$$\int_0^{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2} d\theta = *$$

$$z = re^{i\theta} \Rightarrow |z| = r$$

$$\textcircled{1} \quad |1-z|^2 = (1-z)(1-\bar{z}) = 1 - \bar{z} - z + |z|^2 = 1 - re^{i\theta} - re^{-i\theta} + r^2$$

$$= 1 - 2r\cos\theta + r^2$$

$$\textcircled{2} \quad 1-r^2 = 1-|z|^2$$

$$\textcircled{3} \quad dz = ie^{i\theta} d\theta \Leftrightarrow d\theta = \frac{dz}{iz}$$

$$* = \int_{|z|=r} \frac{1-|z|^2}{1-z^2} \cdot \frac{dz}{iz} = \frac{1}{i} \left( \int_{|z|=r} \left( \sum_{n=0}^{\infty} z^{n-1} + \sum_{n=1}^{\infty} \frac{\bar{z}^n}{z} \right) dz \right) =$$

$$\frac{1}{i} \left( \int_{|z|=r} \frac{dz}{z} + \underbrace{\sum_{n=1}^{\infty} \int_{|z|=r} z^{n-1} dz}_{=0} + \sum_{n=1}^{\infty} \int_{|z|=r} \frac{|z|^n}{z^{n+1}} dz \right) =$$

$$\frac{1}{i} \left( \int_{|z|=r} \frac{dz}{z} + \underbrace{\sum_{n=1}^{\infty} \frac{r^n}{z^{n+1}} dz}_{=0} \right) = \frac{1}{i} \int_{|z|=r} \frac{dz}{z} = \frac{2\pi i}{i} = 2\pi$$

$$6.18 \quad \sum_{k=1}^{\infty} \frac{(-1)^k (x^2 + k)}{k^2}$$

1) Serien är alternverande, ify  $(-1)^k$

2)  $a_k \rightarrow 0$  di  $k \rightarrow \infty$ , ify

$$\left| \frac{(-1)^k (x^2 + k)}{k^2} \right| = \frac{x^2 + k}{k^2} = \frac{x^2}{k^2} + \frac{1}{k} \leq \frac{R^2}{k^2} + \frac{1}{k} \rightarrow 0 \text{ di } k \rightarrow \infty$$

3)  $|a_k|$  är avtagande, ify

$$\left| \frac{(-1)^k (x^2 + k)}{k^2} \right| = \frac{x^2 + k}{k^2} = \frac{x^2}{k^2} + \frac{1}{k} = f(x)$$

$$f'(k) = -\frac{2x^2}{k^3} - \frac{1}{k^2} = -\frac{1}{k^2} \left( \frac{2x^2}{k} - 1 \right) < 0 \text{ för tillräckligt stora } k$$

för något  $x \in [-R, R]$  gäller di att  $\lim_{x \rightarrow \infty} |f_n(x)| \leq \frac{x^2 + n+1}{(n+1)^2}$

Speciellt

$$\left\| \frac{(-1)^k (x^2 + k)}{k^2} \right\| = \sup_{x \in [-R, R]} \left| \frac{(-1)^k (x^2 + k)}{k^2} \right| = \frac{R^2 + k}{k^2} \leq \frac{R^2 + n+1}{(n+1)^2} \rightarrow$$

0 di  $n \rightarrow \infty$

$$|f_n(x)| = \frac{x^2 + k}{k^2} = \left(\frac{x}{k}\right)^2 + \frac{1}{k} \geq \frac{1}{k} \rightarrow 0 \text{ di } k \rightarrow \infty \Rightarrow$$

Serien är inte absolutkonvergent

b.22  $f_n: \mathbb{R} \rightarrow \mathbb{R}$  definieras genom  $f_n(x) = \frac{x^2}{\sqrt{x^2 + 1/n}}$

a)  $\frac{x^2}{\sqrt{x^2 + 1/n}} \rightarrow \frac{x^2}{\sqrt{x^2}} = |x| \text{ e. } n \rightarrow \infty \text{ ty. } \sqrt[n]{x^2} = |x|$

enligt definition,  $x$  är oakt

b)

$$\left| \left| \frac{x^2}{\sqrt{x^2 + 1/n}} - |x| \right| \right| = \sup_{x \in \mathbb{R}} \left| \frac{x^2}{\sqrt{x^2 + 1/n}} - |x| \right| = \frac{1}{\sqrt{1+1/n}} \rightarrow 1$$

O di  $n \rightarrow \infty \Rightarrow$  likformig konvergens för funktionssätet med  $f_n$

c)

$$f'_n = \frac{2x\sqrt{x^2 + 1/n} - x^2 \cdot \frac{1}{\sqrt{x^2 + 1/n}} \cdot x}{x^2 + 1/n} = \frac{2x(x^2 + 1/n) - x^3}{(x^2 + 1/n)^{3/2}} =$$

$$\frac{x^3 - 2x/n}{(x^2 + 1/n)^{3/2}} \rightarrow \frac{x^3}{|x|^3} = \pm 1 \text{ di. } n \rightarrow \infty \Rightarrow \dots$$

punktsvis men ej likformig konvergens