

Kapitel 5

5.1 a) $\sum_{k=0}^5 a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$

b) $\sum_{k=2}^7 a_{k-2} = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$

c) $\sum_{k=0}^5 a_{2k+1} = a_1 + a_3 + a_5 + a_7 + a_9 + a_{11}$

5.2 $1^2 + 4^2 + 9^2 + \dots + (2n)^2 = \sum_{k=1}^n 4k^2 = 4 \sum_{k=1}^n k^2$

5.4

$$1 \times 1 \Rightarrow 0, \quad 2 \times 2 \Rightarrow 1, \quad 3 \times 3 \Rightarrow 3, \quad 4 \times 4 \Rightarrow b$$

$$n \times n \Rightarrow \sum_{k=1}^n k-1 = \sum_{k=1}^n (k) - n$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$= \frac{n(1+n)}{2} - n = \frac{n(n-1)}{2}$$

5.5

$$\sum_{k=0}^n k(k-1)z^{k-2} = 0 + 0 + 2 + 6z + 12z^2 + 20z^3 + \dots + n(n-1)z^{n-2}$$

$$= \sum_{k=0}^{n-2} (k+2)(k+1)z^k$$

$$5.6 \quad \sum_{k=2}^{20} (-2)^k = \frac{(-2)^{21} - (-2)^2}{-2-1} = \frac{4 - (-2)^4}{3} = \frac{4 + 2^4}{3}$$

$$5.7 \quad \sum_{k=1}^n \frac{k-1}{k!} = \sum_{k=1}^n \left(\frac{1}{(k-1)!} - \frac{1}{k!} \right) = - \sum_{k=1}^n \left(\frac{1}{k!} - \frac{1}{(k-1)!} \right) =$$

$$-\left(\frac{1}{n!} - \frac{1}{0!} \right) = 1 - \frac{1}{n!}, \text{ Teleskopanteile}$$

$$5.8 \quad \sum_{k=2}^n \frac{1}{k^{k-1}} = \sum_{k=2}^n \frac{1}{(n+1)(n+1)} \cdot \frac{1}{(n+1)(n+1)} = \frac{A}{n+1} + \frac{B}{n+1} \Rightarrow$$

$$\begin{cases} A+B=0 \Rightarrow A=-B \\ A+B=1 \Rightarrow B=1/2 \Rightarrow A=-\frac{1}{2} \end{cases}$$

$$\sum_{k=2}^n \frac{1}{(n+1)(n+1)} = \frac{1}{2} \cdot \sum_{k=2}^n \left(\frac{1}{n+1} - \frac{1}{n+1} \right) = [f(n) = 1/n] =$$

$$\frac{1}{2} (f(1) - f(3) + f(2) - f(4) + f(3) - f(5) + f(4) - f(6) + \dots + f(n-1) - f(n+1))$$

$$= \frac{1}{2} (f(1) + f(2) - f(n) - f(n+1)) = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \right) =$$

$$\frac{3}{4} - \frac{2n+1}{2n(n+1)} = \frac{3}{4} - \frac{4n+2}{4n(n+1)} = \frac{3}{4} - \frac{2n+2+2n}{2n(2n+2)} =$$

$$\frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$$

5.9 En talfoljd är en ordnad (oändlig) lista av tal
Serier är oändliga summer

5.10 a)

$$\sum_{k=1}^{\infty} (1+i)^k, \quad (1+i)^k \not\rightarrow 0 \text{ då } k \rightarrow \infty \Rightarrow \text{divergent}$$

$$b) \sum_{k=1}^{\infty} (1+i)^{-k}, \quad \sum_{k=1}^{\infty} |(1+i)^{-k}| = \sum_{k=1}^{\infty} \frac{1}{\sqrt[2]{k}} \rightarrow 0 \text{ då } k \rightarrow \infty,$$

dvs är konvergent.

$$\sum_{k=1}^R (1+i)^{-k} = \frac{\left(\frac{1}{1+i}\right)^{R+1} - \frac{1}{1+i}}{\frac{1}{1+i} - 1} =$$

$$\frac{-\frac{1}{1+i}}{\frac{1}{1+i} - 1} \text{ då } R \rightarrow \infty = \frac{\frac{-1}{1+i}}{\frac{-i}{1+i}} = \frac{1}{i} = -i$$

$$c) \sum_{k=1}^{\infty} \frac{i^k}{100}, \quad \frac{i^k}{100} \not\rightarrow 0 \text{ då } k \rightarrow \infty \Rightarrow \text{divergent}$$

$$5.13 \quad S_n = \sum_{k=0}^n a_k = \sqrt{n}$$

$$a_0 = \sqrt{0} = 0, \quad a_0 + a_1 = \sqrt{1} = 1, \quad a_0 + a_1 + a_2 + a_3 + a_4 = \sqrt{5} = 2$$

$$S_n - S_{n-1} = a_n \Leftrightarrow \sqrt{n} - \sqrt{n-1} = a_n \Rightarrow$$

$$a_k = \sqrt{k} - \sqrt{k-1}, \quad a_0 = 0$$

5.15 a) $\sum_{k=1}^{\infty} \cos\left(\frac{1}{k}\right)$, $\cos\left(\frac{1}{k}\right) \rightarrow 1$ då $k \rightarrow \infty \Rightarrow$ divergens

b) $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{\ln k}$, $\frac{k^{1/2}}{\ln k} \rightarrow \infty$ då $k \rightarrow \infty \Rightarrow$ divergens

c) $\sum_{k=1}^{\infty} \frac{(k+1)^2}{3k^2+1}$, $\frac{(k+1)^2}{3k^2+1} = \frac{k^2}{k^2} \left(\frac{1}{3 + \frac{1}{k^2}} + \frac{\frac{2}{k}}{3 + \frac{1}{k^2}} + \frac{\frac{1}{k^2}}{3 + \frac{1}{k^2}} \right)$

$\rightarrow \frac{1}{3}$ då $k \rightarrow \infty \Rightarrow$ divergens

5.16 a) $\sum_{k=2}^{\infty} \frac{2}{k-1}$, $\frac{2}{k-1} \geq \frac{2}{k}$ vars serie divergerar vilket

medfär att $\sum_{k=2}^{\infty} \frac{2}{k-1}$ divergerar

b) $\sum_{k=1}^{\infty} \frac{2^k}{3^k+1}$, $0 \leq \frac{2^k}{3^k+1} \leq \left(\frac{2}{3}\right)^k$, $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ konvergerar \Rightarrow
c konvergerer

c) $\sum_{k=1}^{\infty} \frac{\ln k}{k}$, $\lim_{k \rightarrow \infty} \frac{\ln k}{\frac{1}{k}} = +\infty$, eftersom $\sum_{k=1}^{\infty} \frac{1}{k}$ är divergent
är även c divergent

$$d) \sum_{k=1}^{\infty} \frac{k^8 + 2^k}{3^k - 2^k}, \quad \frac{k^8 + 2^k}{3^k - 2^k} \leq \frac{k^8 + 2^k}{3^k} = \frac{k^8}{3^k} + \left(\frac{2}{3}\right)^k \leq$$

$k^8 \leq 2^k$ för $k \geq 100$ $\Rightarrow 2 \cdot \left(\frac{2}{3}\right)^k, 2 \cdot \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ är konvergent

$\Rightarrow c$ är konvergent

$$e) \sum_{k=1}^{\infty} \sin\left(\frac{1}{k^2}\right), \quad \lim_{k \rightarrow \infty} \frac{\sin(1/k^2)}{1/k^2} = \begin{cases} x = 1/k^2 \\ k \rightarrow \infty \Rightarrow x \rightarrow 0 \end{cases} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ är konvergent $\Rightarrow c$ är konvergent

$$f) \sum_{k=1}^{\infty} \ln(1 + 1/k), \quad \lim_{k \rightarrow \infty} \frac{\ln(1 + 1/k)}{1/k} = \begin{cases} x = 1/k \\ k \rightarrow \infty \Rightarrow x \rightarrow 0 \end{cases} =$$

$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \Rightarrow$ divergent, ty $\sum 1/k$ divergerar

$$g) \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) = - \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}} \right)$$

$$- \sum_{k=1}^R \left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}} \right) = - \left(\frac{1}{\sqrt{R+1}} - \frac{1}{\sqrt{1}} \right) = 1 - \frac{1}{\sqrt{R+1}} \rightarrow 1 \text{ då } R \rightarrow \infty$$

Konvergent, telescoperande

$$24 \quad \sum_{k=1}^n \sin k\theta = \sum_{k=1}^n \operatorname{Im}(e^{ik\theta})$$

$$\sum_{k=1}^n e^{ik\theta} = \sum_{k=1}^n (e^{i\theta})^k = \frac{(e^{i\theta})^{n+1} - e^{i\theta}}{e^{i\theta} - 1} = \frac{i\theta}{2} \frac{(1 - e^{i\theta n})}{1 - e^{i\theta}} \Rightarrow$$

$$\sum_{k=1}^n \sin k\theta = \operatorname{Im}\left(\frac{e^{i\theta}(1 - e^{i\theta n})}{1 - e^{i\theta}}\right) = \operatorname{Im}\left(\frac{e^{i\theta}(1 - e^{i\theta n})}{-e^{i\theta/2}(e^{i\theta n/2} - e^{-i\theta n/2})}\right) =$$

$$\operatorname{Im}\left(\frac{e^{i\theta/2}(1 - e^{i\theta n})}{-2i \sin(\theta/2)}\right) = \operatorname{Im}\left(i \cdot \frac{e^{i\theta/2}(1 - e^{i\theta n})}{2 \sin(\theta/2)}\right) = \operatorname{Re}\left(\frac{e^{i\theta/2}(1 - e^{i\theta n})}{2 \sin(\theta/2)}\right) =$$

$$\operatorname{Re}\left(\frac{e^{i\theta/2} - e^{i\theta(n+1)}}{2 \sin(\theta/2)}\right) = \frac{\cos(\theta/2) - \cos(\theta(n+1))}{2 \sin(\theta/2)} = \frac{\cos(\theta(n+1)) - \cos(\theta/2)}{-2 \sin(\theta/2)}$$

$$= \frac{\cos(\theta n) \cos(\theta/2) - \sin(\theta n) \sin(\theta/2) - \cos(\theta/2)}{-2 \sin(\theta/2)} =$$

$$\frac{\cos(\theta/2)(\cos(\theta n) - 1) - \sin(\theta n) \sin(\theta/2)}{-2 \sin(\theta/2)} =$$

$$\frac{\cos(\theta/2)(\cos^2(\frac{\theta n}{2}) - \sin^2(\frac{\theta n}{2}) - \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2}) - 2 \sin(\frac{\theta n}{2}) \cos(\frac{\theta}{2}) \sin(\frac{\theta}{2}))}{-2 \sin(\theta/2)}$$

$$= \frac{-2 \cos(\theta/2) \sin^2(\theta n/2) - 2 \sin(\theta n/2) \cos(\theta n/2) \sin(\theta/2)}{-2 \sin(\theta/2)} =$$

$$\frac{\sin\left(\frac{\pi n}{2}\right) \left(\cos\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi n}{2}\right) + \cos\left(\frac{\pi n}{2}\right) \sin\left(\frac{\pi}{2}\right) \right)}{\sin\left(\frac{\pi}{2}\right)} =$$

$$\frac{\sin\left(\frac{\pi n}{2}\right) \cdot \sin\left(\frac{(n+1)\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)}$$

5.17 $n\sqrt{n} > 3n \Leftrightarrow n^3 > 9n^2 \Leftrightarrow n^3 - 9n^2 > 0 \Leftrightarrow n^2(n-9) > 0 \Rightarrow n > 9$ för den icke-triviala lösningen
 $9\sqrt{9} = 3 \cdot 9 = 27$, efter $n=9$ stämmer inte likheten i båtfäljden,

ex. $\frac{1}{10\sqrt{10}} = \frac{1}{\sqrt{1000}} < \frac{1}{3 \cdot 10} = \frac{1}{\sqrt{900}}$, eftersom det är seriens "svans" som bestämmer konvergens/divergens källan inte argumentet.

5.18 a) $\sum_{k=2}^{\infty} \underbrace{\frac{1}{k(\ln k)^R}}_{c}$, $\int_2^R \frac{dx}{x\sqrt{\ln x}} = \left[t = \ln x, R \rightarrow \ln R \right] = \int_{\ln 2}^{\ln R} \frac{dt}{\sqrt{t}} =$

$$\left[2\sqrt{t} \right]_{\ln 2}^{\ln R} \rightarrow \infty \text{ då } R \rightarrow \infty \Rightarrow c \text{ divergerar}$$

b) $\sum_{k=2}^{\infty} \underbrace{\frac{1}{k(\ln k)^2}}_c$, $\int_2^R \frac{dx}{x(\ln x)^2} = \left[t = \ln x, R \rightarrow \ln R \right] = \int_{\ln 2}^{\ln R} \frac{dt}{t^2} = \left[-\frac{1}{t} \right]_{\ln 2}^{\ln R} \rightarrow$

$\frac{1}{\ln 2}$ då $R \rightarrow \infty \Rightarrow c$ konvergerar

$$5) \int_{-\infty}^{\infty} x e^{-x^2} dx = \left[t = -x^2, R = -t^2 \right] =$$

$$\int_{-\infty}^{\infty} \frac{1}{2} e^t dt = \left[\frac{e^t}{2} \right]_{-\infty}^{\infty} \rightarrow \frac{1}{2} \text{ da } R \rightarrow \infty \Rightarrow C \text{ konvergent}$$

$$5.19 \sum_{k=1}^n \sqrt{k}$$

$$\int_0^{n+1} \sqrt{x} dx \geq \sum_{k=1}^n \sqrt{k} \geq \int_0^n \sqrt{x} dx \Leftrightarrow$$

$$\frac{2}{3} \left[x^{3/2} \right]_1^{n+1} \geq c \geq \frac{2}{3} \left[x^{3/2} \right]_0^n \Leftrightarrow \frac{2}{3} ((n+1)^{3/2} - 1) \geq c \geq \frac{2}{3} (n^{3/2} - 0)$$

$$\Leftrightarrow \frac{2}{3} ((n+1)^{3/2} - 1) \geq c \geq \frac{2}{3} n^{3/2}$$

$$5.20 \text{ a) } \sum_{k=1}^{\infty} \left| \left(\frac{2k+1}{3^k} \right)^k \right| = \sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{3^k} \cdot \left(\frac{\sqrt[3]{5}}{3} \right)^k \rightarrow 0 \text{ da } k \rightarrow \infty \Rightarrow$$

böde absolutkonvergenz \Leftrightarrow Konvergenz

$$\text{b)} \sum_{k=1}^{\infty} \left| \frac{1}{(2k+1)^2} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{(2k+1)^2}} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \Rightarrow \text{absolut konvergent}$$

c) $\sum_{k=2}^{\infty} \left| \frac{(-1)^k}{\ln k} \right| = \sum_{k=2}^{\infty} \frac{1}{\ln k} \geq \sum_{k=2}^{\infty} \frac{1}{k} \Rightarrow$ absolutdivergens, men

Serien är betingat konvergent enligt Leibniz test ty

- 1. term är arbigraut } $\left\{ \begin{array}{l} \text{serien är också alternerande} \\ 2. a_k \rightarrow 0 \text{ då } k \rightarrow \infty \end{array} \right.$

d) $\sum_{k=1}^{\infty} \left| \frac{\cos 2k}{k^2 + k + 1} \right| \leq \sum_{k=1}^{\infty} \frac{1}{|k^2 + k + 1|} = \sum_{k=1}^{\infty} \frac{1}{(k+\frac{1}{2})^2 + \frac{3}{4}} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \Rightarrow$

absolut \Leftrightarrow vanligt konvergens

e) $\sum_{k=1}^{\infty} \left| \frac{a_{ik}}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2} \Rightarrow$ absolut \Leftrightarrow vanlig konvergens

f) $\sum_{k=1}^{\infty} \left| \frac{1}{k+i} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$, se argument nedan

$$\sum_{k=1}^{\infty} \frac{1}{k+i} = \sum_{k=1}^{\infty} \frac{k-i}{k^2+1} = \sum_{k=1}^{\infty} \frac{k}{k^2+1} - \sum_{k=1}^{\infty} \frac{i}{k^2+1}$$

$$\sum_{k=1}^R \frac{k}{k^2+1} \geq \int_1^{R+1} \frac{x dx}{x^2+1} = \left[t = x^2+1, dt = 2x dx, \frac{dt}{2} = x dx \right] = \frac{1}{2} \int_1^{(R+1)^2+1} \frac{dt}{t} = \frac{1}{2} \left[\ln t \right] \rightarrow \infty \text{ då}$$

$R \rightarrow \infty \Rightarrow$ serien båda är divergent \Leftrightarrow absolutdivergent

g) $\sum_{k=1}^{\infty} |(-1)^k \ln k| = \sum_{k=1}^{\infty} \ln k \Rightarrow$ absolutdivergens

Även vanligt divergent ty $a_k \not\rightarrow 0$ då $k \rightarrow \infty$

w) $\sum_{k=1}^{\infty} \left| \frac{e^{i \sin^2 k}}{2^k} \right| = \sum_{k=1}^{\infty} \frac{1}{2^k} \Rightarrow$ absolut konvergent \Leftrightarrow ~~variig/konvergens~~
 (geometrisk serie)

i) $\sum_{k=1}^{\infty} \frac{1}{\cos k} = \sum_{k=1}^{\infty} \frac{1}{\frac{e^{ik} + e^{-ik}}{2}} = \sum_{k=1}^{\infty} \frac{1}{\cos k}$

$\frac{1}{\cos k} \not\rightarrow 0$ da $k \rightarrow \infty \Rightarrow$ absolut \Leftrightarrow variig divergens

5.2i a) $\sum_{k=0}^{\infty} \frac{10^k}{k!}$

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{10^{k+1}}{(k+1)!}}{\frac{10^k}{k!}} = \frac{10}{k+1} \rightarrow 0 \text{ da } k \rightarrow \infty \Rightarrow \text{absolut konvergenz}$$

b) $\sum_{k=1}^{\infty} \frac{k!}{k^k} \cdot \frac{\frac{(k+1)!}{(k+1)^{k+1}}}{\frac{k!}{k^k}} = \frac{(k+1)k^k}{(k+1)^{k+1}(k+1)} = \frac{k^k}{(k+1)^k} = \left(\frac{k}{k+1}\right)^k =$

$$\left(\frac{1}{1+\frac{1}{k}}\right)^k \rightarrow \frac{1}{e} \text{ da } k \rightarrow \infty \Rightarrow \text{absolut konvergenz}$$

c) $\sum_{k=0}^{\infty} \left(1 - \frac{1}{k}\right)^{k^2}, \quad \left(\left(1 - \frac{1}{k}\right)^k\right)^{1/k} = \left(1 - \frac{1}{k}\right)^k = \left(\left(1 + \frac{1}{-k}\right)^{-k}\right)^{-1}$

$$= \left[\begin{matrix} t = -k \\ k \rightarrow \infty \Rightarrow t \rightarrow -\infty \end{matrix} \right] = \left(\left(1 + \frac{1}{t}\right)^t\right)^{-1} \rightarrow \frac{1}{e} \text{ da } t \rightarrow -\infty$$

$$5.22 \quad \sum_{k=1}^{\infty} (-1)^k k^{-\alpha}$$

Leibniz-qer konvergens om $\alpha > 0$ (antagnit $\epsilon \rightarrow 0$)
 Absolutkonvergens om $\alpha > 1$ (konvergent p-serie)

5.23 a)

$$\sum_{k=1}^{\infty} \frac{z^k}{k^2(k+1)} \quad , \quad \left| \frac{z^k}{k^2(k+1)} \right|^{\frac{1}{k}} = \frac{|z|}{(k^{1/k})^2(k+1)^{1/k}} \leq \frac{|z|}{(k^{1/k})^3} \rightarrow |z| \text{ då } k \rightarrow \infty$$

Om $|z| < 1 \Rightarrow$ absolutkonvergens, $|z| > 1 \Rightarrow$ divergens

Om $|z|=1$ för vi

$$\sum_{k=1}^{\infty} \left| \frac{z^k}{k^2(k+1)} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ som konvergerar} \Rightarrow |z| \leq 1$$

b) $\sum_{k=1}^{\infty} k z^{(k)}, \quad |k z^{(k)}| = k |z|^{(k)} \leq k |z|^k \rightarrow 0$ då $k \rightarrow \infty$ om

$|z| < 1$, dvs $\sum_{k=1}^{\infty} k z^{(k)}$ konvergerar om $|z| < 1$, samt diver-

erar enligt samma argument om $|z| \geq 1$

c) $\sum_{k=1}^{\infty} \left(z - \frac{1}{k} \right)^k, \quad \left| \left(z - \frac{1}{k} \right)^k \right| = \left| z - \frac{1}{k} \right|^k$

Om $|z| = r_1 < 1 \Rightarrow \left| z - \frac{1}{k} \right| < r_1 + \varepsilon < 1$ där ε är
 tillräckligt litet för stora k .

Om $|z| = r_2 \geq 1$ kan $\left| z - \frac{1}{k} \right| > r_2 + \varepsilon > 1$, om ex. $z = -1$

Därav måste $|z| < 1$

$$5.24 \quad S = \sum_{k=1}^n \frac{2^k}{k!} \quad , \quad \left| \frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} \right| = \frac{2}{k+1} \rightarrow 0 \text{ as } k \rightarrow \infty$$

$$|S - S_{10}| = |r_{10}| = \sum_{k=11}^n \frac{2^k}{k!} = \frac{2^n}{n!} + \frac{2^{12}}{12!} + \dots =$$

$$\frac{2^n}{n!} \left(1 + \frac{2}{12} + \frac{2^2}{12 \cdot 13} + \dots \right) \leq \frac{2^n}{n!} \left(1 + \left(\frac{2}{12}\right)^1 + \left(\frac{2}{12}\right)^2 + \dots \right) =$$

$$\frac{2^n}{n!} \sum_{k=0}^n \frac{1}{6^k} = \frac{2^n}{n!} \cdot \frac{\frac{1}{6^{n+1}} - 1}{\frac{1}{6} - 1} = \frac{2^n}{n!} \cdot \frac{6}{5} \cdot \left(1 - \frac{1}{6^{n+1}} \right) \leq \frac{6}{5} \cdot \frac{2^n}{n!}$$

$$5.25 \text{ a)} \quad S = \sum_{k=1}^{\infty} k^{-3}, \quad S_{10} = \sum_{k=1}^{10} k^{-3}$$

$$|S - S_{10}| = |r_{10}| = \sum_{k=11}^{\infty} k^{-3} \leq \int_{10}^{\infty} \frac{dx}{x^3} = \left[-\frac{1}{2x^2} \right]_{10}^{\infty} = \frac{1}{2 \cdot 100} = 0,005$$

$$|S - S_n| = \frac{1}{2 \cdot n^2} \leq 10^{-6} \Leftrightarrow n^2 \geq \frac{10^6}{2} \Leftrightarrow n \geq \frac{1000}{\sqrt{2}} = 707$$

$$\text{b)} \quad S = \sum_{k=1}^{\infty} (-1)^k k^{-3}, \quad S_{10} = \sum_{k=1}^{10} (-1)^k k^{-3}$$

$$|S - S_{10}| = |r_{10}| = \sum_{k=11}^{\infty} (-1)^k k^{-3} \leq \left| \frac{(-1)^n}{11^3} \right| \leftarrow 0,001$$

a_n uppfyller Leibniz test

$$|S - S_n| \leq \frac{1}{(n+1)^3} < 10^{-6} \Leftrightarrow (n+1)^3 > 10^6 \Leftrightarrow n > 100 - 1 = 99$$

5.28 Härled en formel för $\sum_{k=0}^n kx^k$

$$f(x) = \sum_{k=0}^n x^k = \frac{x^{n+1}-1}{x-1}$$

$$\textcircled{1} \quad f'(x) = \sum_{k=0}^n kx^{k-1} = \frac{1}{x} \sum_{k=0}^n kx^k$$

$$\textcircled{2} \quad f'(x) = \frac{(n+1)x^n(x-1) - (x^{n+1}-1)}{(x-1)^2} = \frac{x^n(nx-n+x-1) - x^{n+1} + 1}{(x-1)^2} =$$

$$\frac{x^n(nx-n-1) - 1}{(x-1)^2}$$

$$\textcircled{1} = \textcircled{2} \iff \sum_{k=0}^n kx^k = \frac{x^{n+1}(nx-n-1) + 1}{(x-1)^2}$$

$$\text{Om } x=1 \Rightarrow f(x)=f(1) = \sum_{k=0}^n k = \frac{n(n+1)}{2} \quad (\text{aritmetisk})$$

$$5.31 \quad \sum_{k=0}^{\infty} (-1)^k a_k, \quad 0 < a_{k+1} < a_k$$

- 1) Serien är alternerande
- 2) a_k är avtagande

om $a_k \neq 0$ då $k \rightarrow \infty$ kan serien divergera

$$a_k = \frac{1}{k} + 1 \rightarrow 1 \text{ då } k \rightarrow \infty$$

5.33 a) $\sum_{k=0}^{\infty} a_k$ är konvergent

$a_k = \frac{(-1)^k}{\sqrt{k}}$ är konvergent men $a_k^2 = \frac{1}{k}$ är det inte

b) Ja, då kan ibbe serien vara alternerande
 för stora k :n gäller då $a_k < 1$, ty $a_k \rightarrow 0 \Rightarrow$
 $0 < a_k^2 < a_k \Rightarrow a_k^2$ är konvergent

5.45 För vilka reella värden på α konvergerar serien

$$\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)^{\alpha}$$

$$\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)^{\alpha} = \left(\frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k(k+1)}} \right)^{\alpha} = \left(\frac{(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k})}{(\sqrt{k+1} + \sqrt{k})\sqrt{k(k+1)}} \right)^{\alpha} = \\ \left(\frac{1}{k^{1/2} \sqrt{1 + \frac{1}{k}}} \cdot k^{1/2} \cdot k^{1/2} \sqrt{1 + \frac{1}{k}} \right)^{\alpha} \leq \left(\frac{1}{k^{3/2}} \right)^{\alpha} \Rightarrow \alpha > 2/3 \text{ för att det ska bli en konvergent p-serie}$$