

Kapitel 4

$$4.1 \quad a_0 = 41, \quad a_n = 41 - 4n \Rightarrow a_{22} = 41 - 4 \cdot 22 = -47$$

$$4.2 \text{ a) } k = \frac{a_{n+1}}{a_n} = \frac{a_5}{a_4} = \frac{-3}{6} = -\frac{1}{2}$$

$$a_n = a_1 \cdot \left(-\frac{1}{2}\right)^{n-1} \Rightarrow a_0 = \frac{a_4}{\left(-\frac{1}{2}\right)^3} = \frac{6}{\left(-\frac{1}{8}\right)} = -48$$

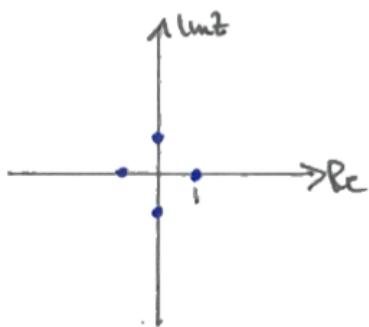
$$\text{b) } a_4 = 6, \quad a_6 = -3$$

$$k = \frac{a_5}{a_4} = \frac{a_6}{a_5} \Leftrightarrow a_5^2 = a_6 \cdot a_4 = -18 \Rightarrow a_5 = \sqrt{-18} = \pm 3i\sqrt{2}$$

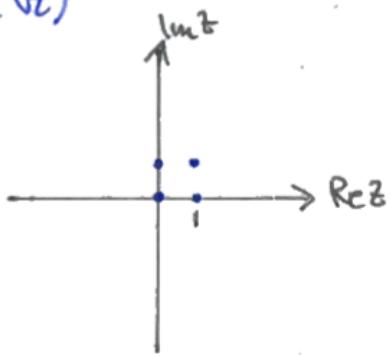
$$k = \pm \frac{3i\sqrt{2}}{6} = \pm \frac{i\sqrt{2}}{2} = \pm \frac{i}{\sqrt{2}}$$

$$a_4 = a_1 \cdot \left(\pm \frac{i}{\sqrt{2}}\right)^3 \Rightarrow a_1 = \frac{6}{\left(\pm \frac{i}{\sqrt{2}}\right)^3} = \frac{12\sqrt{2}}{\pm i} = \pm 12\sqrt{2}$$

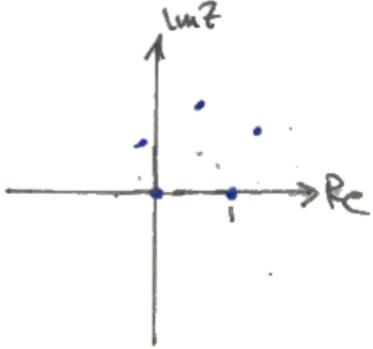
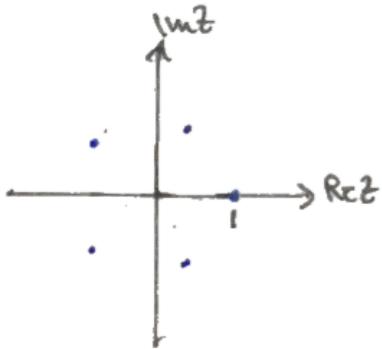
4.3 a)



b)



c)



4.4 a)

$$\lim_{n \rightarrow \infty} n^{-n} = \lim_{n \rightarrow \infty} e^{-n \ln(n)} = 0$$

b)

$$\lim_{n \rightarrow \infty} n^{-1/n} = \lim_{n \rightarrow \infty} e^{-\frac{\ln n}{n}} = 1$$

c) $\lim_{n \rightarrow \infty} \frac{2^n}{3^n + 1} = \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n \cdot \frac{1}{1 + \frac{1}{3^n}} = 0$

d) $\lim_{n \rightarrow \infty} \frac{(-1)^n}{2^n + 2^{-n}} = \lim_{n \rightarrow \infty} \left(\frac{-1}{2}\right)^n \cdot \frac{1}{1 + \frac{1}{2^{2n}}} = 0$

4.7 a) $x_{n+1} + x_n = 0$, $x_n = Cr^n$

$$x_{n+1} + x_n = 0 \Leftrightarrow Cr^{n+1} + Cr^n = 0 \Leftrightarrow r+1=0 \Leftrightarrow r=-1$$

$$x_n = C \cdot (-1)^n$$

b) $x_{n+1} - 3x_n = 0 \Leftrightarrow r-3=0 \Leftrightarrow r=3$ om $x_n = Cr^n$

$$x_n = C3^n$$

c) $x_{n+2} + 5x_{n+1} + 4x_n = 0 \Leftrightarrow r^2 + 5r + 4 = 0 \Leftrightarrow \left(r + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{16}{4} = 0$

$$\Leftrightarrow \left(r + \frac{5}{2}\right)^2 = \frac{9}{4} \Leftrightarrow r = \pm \frac{3}{2} - \frac{5}{2} \Leftrightarrow r_1 = -1, r_2 = -4$$

$$x_n = C_1 (-1)^n + C_2 (-4)^n$$

$$d) \quad x_{n+2} - 4x_{n+1} + 4x_n = 0 \Leftrightarrow r^2 - 4r + 4 = 0 \Leftrightarrow$$

$$(r-2)^2 = 0 \Leftrightarrow r_{1,2} = 2$$

$$x_n = (c_1 + c_2 n) 2^n$$

$$e) \quad x_{n+2} + 4x_{n+1} + 5x_n = 0 \Leftrightarrow r^2 + 4r + 5 = 0 \Leftrightarrow$$

$$(r+2)^2 = i^2 \Leftrightarrow r = -2 \pm i$$

$$x_n = c_1 (-2+i)^n + c_2 (-2-i)^n$$

$$4.8 \text{ a)} \quad x_{n+1} + x_n = 0, \quad x_0 = 2$$

$$r+1=0 \Rightarrow r=-1 \Rightarrow x_n = c (-1)^n$$

$$x_0 = c = 2 \Rightarrow x_n = 2(-1)^n$$

$$b) \quad x_{n+1} - 3x_n = 2, \quad x_0 = 1$$

$$r-3=0 \Leftrightarrow r=3 \Rightarrow x_n = c 3^n$$

$$x_n^P = A \Rightarrow A - 3A = 2 \Leftrightarrow A = -1$$

$$x_n = c 3^n - 1$$

$$x_0 = 1 \Rightarrow 1 = c - 1 \Leftrightarrow c = 2 \Rightarrow x_n = 2 \cdot 3^n - 1$$

$$c) \quad x_{n+2} + 5x_{n+1} + 4x_n = 10n, \quad x_0 = 1, \quad x_1 = -2$$

$$r^2 + 5r + 4 = 0 \Leftrightarrow r_1 = -1, \quad r_2 = -4 \Rightarrow x_n^h = c_1 (-1)^n + c_2 (-4)^n$$

$$x_n^P = An + B \Rightarrow A(n+2) + B + 5(A(n+1) + B) + 4(An + B) = 10n$$

$$\Rightarrow \begin{cases} 10A = 10 \Leftrightarrow A = 1 \\ 7A + 10B = 0 \Rightarrow B = -7/10 \end{cases}$$

$$x_n^P = n - 7/10$$

$$x_n = c_1 (-1)^n + c_2 (-4)^n + n - 7/10$$

$$x_0 = c_1 + c_2 - 7/10 = 1 \Leftrightarrow c_1 + c_2 = 17/10$$

$$x_1 = -c_1 - 4c_2 + 1 - 7/10 = -2 \Leftrightarrow c_1 + 4c_2 = 23/10$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 17/10 \\ c_1 + 4c_2 = 23/10 \end{cases} \Leftrightarrow 3c_2 = 6/10 \Leftrightarrow c_2 = 1/5$$

$$\Rightarrow c_1 = 17/10 - 2/10 = 15/10 = 3/2$$

$$x_n = \frac{3}{2} (-1)^n + \frac{1}{5} (-4)^n + n - \frac{7}{10}$$

$$d) \quad x_{n+2} + 5x_{n+1} + 4x_n = (-1)^n, \quad x_0 = 0, \quad x_1 = 2$$

$$r^2 + 5r + 4 = 0 \Leftrightarrow r_1 = -1, \quad r_2 = -4$$

$$x_n^P = c_1 (-1)^n + c_2 (-4)^n$$

Aussetzung $x_n^P = a(-1)^n$ f\u00f6rgerne m\u00f6glich, da \u00e4tzt funktion $x_n^P = an(-1)^n$

$$a(n+2)(-1)^{n+2} + 5a(n+1)(-1)^{n+1} + 4an(-1)^n = (-1)^n \Leftrightarrow$$

$$(-1)^n (a(n+2) - 5a(n+1) + 4an) = (-1)^n \Leftrightarrow -3a = 1 \Leftrightarrow a = -1/3$$

$$x_n^P = -\frac{n}{3}(-1)^n \Rightarrow x_n = c_1 (-1)^n + c_2 (-4)^n - \frac{n}{3}(-1)^n$$

$$x_0 = 0 \Rightarrow c_1 + c_2 = 0 \Leftrightarrow c_1 = -c_2$$

$$x_1 = 2 \Rightarrow -c_1 - 4c_2 + \frac{1}{3} = 2 \Leftrightarrow 3c_1 = 5/3 \Leftrightarrow c_1 = 5/9 \Rightarrow c_2 = -5/9$$

$$x_n = \frac{5}{9}((-1)^n - (-4)^n) - \frac{n}{3}(-1)^n$$

e) $x_{n+2} - 4x_{n+1} + 4x_n = n \cdot 3^n, x_0 = 4, x_1 = 1$

$$r^2 - 4r + 4 = 0 \Leftrightarrow (r-2)^2 = 0 \Leftrightarrow r_{1,2} = 2$$

$$x_n = (c_1 + c_2 n) 2^n$$

Ansätzen $x_n^P = (a + bn) 3^n$

$$(a + b(n+2)) 3^{n+2} - 4(a + b(n+1)) 3^{n+1} + 4(a + bn) 3^n = n \cdot 3^n \Leftrightarrow$$

$$9(a + b(n+2)) - 12(a + b(n+1)) + 4(a + bn) = n \Leftrightarrow$$

$$\begin{cases} a + bb = 0 \Leftrightarrow a = -b \\ b = 1 \end{cases}$$

$$x_n^P = (n - b) 3^n \Rightarrow x_n = (c_1 + c_2 n) 2^n + (n - b) 3^n$$

$$x_0 = 4 \Rightarrow c_1 - b = 4 \Leftrightarrow c_1 = 10$$

$$x_1 = 1 \Rightarrow (10 + c_2) 2 - 5 \cdot 3 = 1 \Leftrightarrow c_2 = -2$$

$$x_n = (10 - 2n) 2^n + (n - b) 3^n$$

$$f) \quad x_{n+2} - 4x_{n+1} + 4x_n = 2^n, \quad x_0 = 0, \quad x_1 = 1$$

$$r^2 - 4r + 4 \Rightarrow r_{1/2} = 2 \Rightarrow x_n^h = (c_1 + c_2 n) 2^n$$

$$\text{Ansatz } x_n^p = a n^2 2^n$$

$$a(n+2)^2 \cdot 2^{n+2} - 4a(n+1)^2 2^{n+1} + 4an^2 \cdot 2^n = 2^n \Leftrightarrow$$

$$4a(n^2 + 4n + 4) - 8a(n^2 + 2n + 1) + 4an^2 = 1 \Leftrightarrow 8a = 1 \Leftrightarrow a = 1/8$$

$$x_n^p = \frac{n^2}{8} 2^n \Rightarrow x_n = (\frac{n^2}{8} + c_2 n + c_1) \cdot 2^n$$

$$x_0 = 0 \Rightarrow c_1 = 0$$

$$x_1 = 1 \Rightarrow 2(\frac{1}{8} + c_2) = 1 \Leftrightarrow c_2 = 3/8$$

$$x_n = (n^2 + 3n) \frac{2^n}{8}$$

$$410 \quad S_n = \sum_{k=1}^n k^2$$

$$a) \quad S_{n+1} - S_n = (n+1)^2$$

$$b) \quad r - 1 = 0 \Leftrightarrow r = 1 \Rightarrow x_n^h = c_1$$

$$x_n^p = (an^2 + bn + c)n = an^3 + bn^2 + cn$$

$$a(n+1)^3 + b(n+1)^2 + c(n+1) - (an^3 + bn^2 + cn) = (n+1)^2 \Leftrightarrow$$

$$a(n^3 + 3n^2 + 3n + 1) + b(n^2 + 2n + 1) + c(n+1) - an^3 - bn^2 - cn = n^2 + 2n + 1 \Leftrightarrow$$

$$\left\{ \begin{array}{l} 3a = 1 \Leftrightarrow a = 1/3 \\ 3a + 2b = 2 \Leftrightarrow b = 1/2 \end{array} \right.$$

$$a + b + c - 1 \Leftrightarrow c = b/6 - 2/6 - 3/6 = 1/6$$

$$x_n^p = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \Rightarrow x_n = C_1 + \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$x_1 = 1 \Rightarrow C_1 + \frac{1}{6} + \frac{3}{6} + \frac{1}{6} = 1 \Leftrightarrow C_1 = 0$$

$$x_n = \frac{1}{6}(2n^3 + 3n^2 + n) = \frac{n}{6}(2n^2 + 3n + 1) = \frac{n}{6}(2n+1)(n+1)$$

4.11 a) $x_{n+2} = x_{n+1} - x_n$

$$r^2 - r + 1 = 0 \Leftrightarrow (r - \frac{1}{2})^2 + \frac{3}{4} = 0 \Leftrightarrow r = \pm \frac{\sqrt{3}i}{2} + \frac{1}{2} = \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

$$r = e^{\pm i\pi/3} \Rightarrow x_n = C_1 e^{i\pi n/3} + C_2 e^{-i\pi n/3} =$$

$e^{\pm i\pi/3}$ är bilden periodiska med period 6, ty $\frac{2\pi}{\pi/3} = 6$

$$x_{n+1} = C_1 e^{i\pi(n+6)/3} + C_2 e^{-i\pi(n+6)/3} = x_n$$

b) $x_0 = 2, x_1 = -3$

$$x_0 = 2 \Rightarrow C_1 + C_2 = 2 \Leftrightarrow C_1 = 2 - C_2$$

$$x_1 = -3 \Rightarrow C_1 e^{i\pi/3} + C_2 e^{-i\pi/3} = -3 \Leftrightarrow (2 - C_2) e^{i\pi/3} + C_2 e^{-i\pi/3} = -3 \Leftrightarrow$$

$$2e^{i\pi/3} - C_2 e^{i\pi/3} + C_2 e^{-i\pi/3} = -3 \Leftrightarrow 1 + i\sqrt{3} - 2iC_2 \sin(\pi/3) = -3 \Leftrightarrow$$

$$1 + i\sqrt{3} - i\sqrt{3} C_2 = -3 \Leftrightarrow i\sqrt{3} C_2 = 4 + i\sqrt{3} \Leftrightarrow C_2 = 1 - \frac{i4}{\sqrt{3}} \Rightarrow$$

$$C_1 = 1 + \frac{i4}{\sqrt{3}} \Rightarrow x_n = \left(1 + \frac{i4}{\sqrt{3}}\right) e^{i\pi n/3} + \left(1 - \frac{i4}{\sqrt{3}}\right) e^{-i\pi n/3}$$

$$1000 = b \cdot 166 + 4 \Rightarrow x_{1000} = x_4 = \left(1 + \frac{i4}{\sqrt{3}}\right) e^{4i\pi/3} + \left(1 - \frac{i4}{\sqrt{3}}\right) e^{-4i\pi/3} =$$

$$\left(1 + \frac{i4}{\sqrt{3}}\right) \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) + \left(1 - \frac{i4}{\sqrt{3}}\right) \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = -\frac{1}{2} \left(1 + i\sqrt{3} + \frac{4i}{\sqrt{3}} - 4\right) +$$

$$\frac{1}{2} \left(-1 + i\sqrt{3} + \frac{4i}{\sqrt{3}} + 4\right) = \frac{3}{2} + \frac{3}{2} = 3$$

4.12 a) Tank logistet:)

b) $x_{n+1} = 2x_n + 1$

$$r = 2 \Rightarrow x_n = A2^n$$
$$x_0^p = a \Rightarrow a = 2a + 1 \Leftrightarrow a = -1 \quad \left. \begin{array}{l} x_n = A2^n - 1 \end{array} \right\}$$

$$x_0 = 0 \Rightarrow 0 = A - 1 \Rightarrow A = 1 \Rightarrow x_n = 2^n - 1$$

4.13 $x_n = nx_{n-1}, x_1 = 1$

$$x_2 = 2 \cdot x_1 = 2 \cdot 1, x_3 = 3 \cdot x_2 = 3 \cdot 2 = 6, x_4 = 4 \cdot 6 = 24$$

$$x_n = n! \Rightarrow x_{10} = 10!$$

4.9 $x_{n+2} + ax_{n+1} + bx_n = 0, x_0 = 1, x_1 = 3, x_2 = 7, x_3 = 15$

$$r^2 + ar + b = 0 \Leftrightarrow (r + \frac{a}{2})^2 - \frac{a^2}{4} + b = 0 \Leftrightarrow (r + \frac{a}{2})^2 = \frac{a^2}{4} - b \Leftrightarrow$$

$$r = \pm \sqrt{\frac{a^2}{4} - b} - \frac{a}{2}, \quad x_n = Ar_1^n + Br_2^n$$

$$x_0 = 1 \Rightarrow A + B = 1 \Rightarrow B = 1 - A$$

$$x_1 = 3 \Rightarrow Ar_1 + Br_2 = 3 \Rightarrow Ar_1 + (1 - A)r_2 = 3$$

$$\begin{aligned} ① \quad & \left\{ \begin{array}{l} 7 + 3r_1 + b = 0 \\ 15 + 7r_2 + 3b = 0 \end{array} \right. \Leftrightarrow -b - 2r_2 = 0 \Leftrightarrow b = -2r_2 \\ ② \quad & \left. \begin{array}{l} 7 + 3r_1 + b = 0 \\ 15 + 7r_2 + 3b = 0 \end{array} \right. \Leftrightarrow -b - 2r_1 = 0 \Leftrightarrow b = -2r_1 \end{aligned}$$

$$r = \pm \sqrt{\frac{(-3)^2}{4} - 2} - \frac{(-3)}{2} = \pm \frac{1}{2} + \frac{3}{2} \Rightarrow r_1 = 2, r_2 = 1$$

$$2A + (1 - A) = 3 \Leftrightarrow A = 2 \Rightarrow B = -1 \Rightarrow x_n = 2^{n+1} - 1$$

$$x_{100} = 2^{101} - 1$$