

Kapitel 3

$$3.1 \quad \int_0^{\pi} \cos it \, dt = \left[\frac{\sin it}{i} \right]_0^{\pi} = \frac{\sin \pi i}{i} = \frac{e^{-\pi} - e^{\pi}}{-2} = \frac{1}{2} \cdot (e^{\pi} - e^{-\pi})$$

$$3.2 \quad \int_0^1 \frac{dt}{1-it} = \int_0^1 \frac{1+it}{1+t^2} dt = \int_0^1 \frac{dt}{1+t^2} + i \int_0^1 \frac{t dt}{1+t^2} =$$

$$\left[\operatorname{arctant} \right]_0^1 + \frac{i}{2} \left[\ln(1+t^2) \right]_0^1 = \frac{\pi}{4} + \frac{i \ln 2}{2}$$

$$3.3 \text{ a) } |z+1| = 3, \quad z(t) = -1 + 3(\cos t + i \sin t) =$$

$$-1 + 3e^{it}, \quad 0 \leq t \leq 2\pi$$

$$\text{b) } z(t) = -1 + 3e^{it}, \quad t: 0 \rightarrow -8\pi$$

ab.

$$z(t) = -1 + 3e^{-it}, \quad 0 \leq t \leq 2\pi$$

3.4 a)

$$\int_{\gamma} z dz, \quad \gamma = x^2 \quad (-1,1) \rightarrow (1,1) \Rightarrow z(t) = t + it^2, \quad -1 \leq t \leq 1$$

$$dz = (1 + 2it) dt$$

$$\int_{\gamma} z dz = \int_{-1}^1 (t + it^2)(1 + 2it) dt = \int_{-1}^1 (t + 2it^2 + it^2 - 2t^3) dt =$$

$$\left[\frac{t^2}{2} + \frac{2it^3}{3} + \frac{it^3}{3} - \frac{t^4}{4} \right]_{-1}^1 = \left[\frac{t^2}{2} + it^3 - \frac{t^4}{4} \right]_{-1}^1 = i - (-i) = 2i$$

$$f(z) = z, \quad (-1+i) \rightarrow (1+i)$$

$$\begin{aligned} \text{b)} \quad \int_{\gamma} z dz &= F(1+i) - F(-1+i) = \frac{1}{2}((1+i)^2 - (-1+i)^2) \\ &= \frac{1}{2}(1+2i-1 - (1-2i-1)) = \frac{4i}{2} = 2i \end{aligned}$$

$$3.5 \quad \gamma: a=2+i \rightarrow b=i$$

$$\text{a)} \quad z(t) = -t+i, \quad -2 \leq t \leq 0$$

$$\text{b)} \quad f(z) = \operatorname{Re} z = -t \Rightarrow dz = -dt$$

$$\int_{\gamma} \operatorname{Re} z(t) dz = \int_{-2}^0 -t \cdot -dt = \int_{-2}^0 t dt = \frac{1}{2} [t^2]_{-2}^0 = -2$$

c) Nein, by $\operatorname{Re} z$ in der holomorph neighborhood

$$3.6 \quad \int_{\gamma} \frac{dz}{z}$$

$$\text{a)} \quad z(t) = \cos t + i \sin t, \quad \pi \leq t \leq 2\pi, \quad dz = (-\sin t + i \cos t) dt$$

$$\begin{aligned} \int_{\gamma} \frac{dz}{z} &= \int_{\pi}^{2\pi} \frac{(-\sin t + i \cos t) dt}{\cos t + i \sin t} = \int_{\pi}^{2\pi} (-\sin t + i \cos t)(\cos t - i \sin t) dt \\ &= \int_{\pi}^{2\pi} (-\sin t \cos t + i \sin^2 t + i \cos^2 t + \cos t \sin t) dt = \int_{\pi}^{2\pi} i dt \\ &= \left[it \right]_{\pi}^{2\pi} = \pi i \end{aligned}$$

, $K \in \mathbb{C}$

$$f(z) = \frac{1}{z} \Rightarrow F(z) = \log z + K, \text{ c} \text{ j enbrydigt beständ}$$

Fixera omvända bexasgrenen, def. $-\frac{3\pi}{2} < \arg z < \frac{\pi}{2}$

$$F(z) = \log z = \ln|z| + i \arg(z)$$

$$\int_{\gamma} \frac{dz}{z} = \log(1) - \log(-1) = i \arg(1) - i \arg(-1) = 0 - i(-\pi) = \pi i$$

b) $z(t) = \cos t + i \sin t$, $t: \pi \rightarrow 0$

$$\int_{\gamma} \frac{dz}{z} = \int_{\pi}^0 i dt = \left[it \right]_{\pi}^0 = -\pi i$$

$$f(z) = 1/z \Rightarrow F(z) = \log z + K, K \in \mathbb{C}$$

Fixera bexasgrenen, $-\pi/2 < \arg z < 3\pi/2$

$$\int_{\gamma} \frac{dz}{z} = \log(1) - \log(-1) = i \arg(1) - i \arg(-1) = 0 - i\pi = -\pi i$$

3.9 a) $|f(z)| \leq M$ for $|z| \leq r$, $r < 1$

$$\left| \int_{|z|=r} \frac{f(z)}{z} dz \right| \leq \int_{|z|=r} \left| \frac{f(z)}{z} \right| |dz| \leq \int_{|z|=r} \frac{M |dz|}{r} \leq \frac{M \ell(|z|=r)}{r} =$$

$$\frac{M \cdot 2\pi r}{r} = 2\pi M$$

b) $\left| \int_{|z|=r} z f(z) dz \right| \leq r M \int_{|z|=r} |dz| \leq r M \ell(|z|=r) = 2\pi r^2 M$

$$3.10 \quad \gamma: i \rightarrow 1, \quad f(z) = \frac{1}{z^4}$$

$$z(t) = t + i(1-t), \quad 0 \leq t \leq 1$$

$$|z(t)| = \sqrt{t^2 + (1-t)^2} = \sqrt{2t^2 - 2t + 1}$$

$$|z(t)|' = \frac{1}{2} \cdot \frac{1}{2t^2 - 2t + 1} \cdot (4t - 2)$$

$$|z(t)|' = 0 \Leftrightarrow t = 1/2 \quad (\text{min})$$

$$|z(t)| \geq \sqrt{2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 1} \geq \frac{1}{\sqrt{2}}$$

$$|f(t)| \leq \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^4} \leq 4$$

$$\ell(\gamma) = \sqrt{1^2 + 1^2} = \sqrt{2} \quad (\text{hypotenuse})$$

$$\left| \int_{\gamma} \frac{dz}{z^4} \right| \leq \int_{\gamma} \frac{|dz|}{|z|^4} \leq 4 \int_{\gamma} |dz| \leq 4 \cdot \ell(\gamma) = 4\sqrt{2}$$

$$3.7 \quad \int_{|z|=1} z dz, \quad z(t) = e^{it}, \quad 0 \leq t \leq 2\pi, \\ z'(t) = ie^{it}$$

$$\int_{\gamma} f(z) dz = \int_0^{2\pi} e^{it} \cdot |ie^{it}| dt = \int_0^{2\pi} e^{it} dt = \frac{1}{i} \left[e^{it} \right]_0^{2\pi} =$$

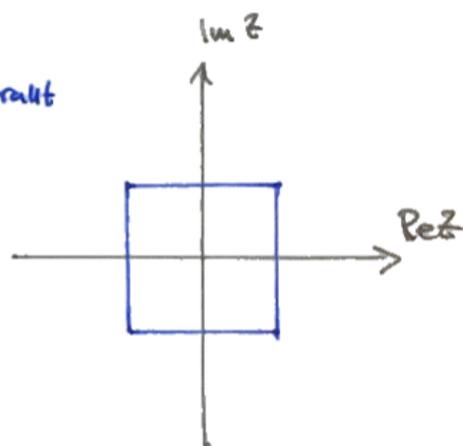
$$\frac{1}{i} (e^{2\pi i} - 1) = 0$$

$$3.11 \quad \int_{|z|=1} \sin z^2 dz = 0, \quad \text{by } \sin z^2 \text{ holomorph}$$

$$3.12 \quad a) \quad \int_{\gamma} z^2 dz = 0, \quad \text{by } z^2 \text{ holomorph overall}$$

$$b) \quad f(z) = \frac{\cos z}{z^2 + 8}, \quad f(0) = 1/8$$

$$\int_{\gamma} \frac{\cos z}{z(z^2 + 8)} = i2\pi \cdot f(0) = i2\pi/8 = i\pi/4$$



$$c) \quad \int_{\gamma} \frac{z dz}{z^2 - 1} = \int_{\gamma} \frac{z/2}{z - 1/2} = \left[f(z) = \frac{z}{2}, f(1/2) = 1/4 \right] =$$

$$= 2\pi i \cdot f(1/2) = i\pi/2$$

$$d) \int_{\gamma} \frac{\sin z \, dz}{z^2 + 1} = \int_{\gamma} \frac{\sin z \, dz}{(z-i)(z+i)} = *$$

$$\frac{1}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i} \Rightarrow (A+B)z + i(A-B) = 1 \Rightarrow$$

$$\begin{cases} A+B=0 \Leftrightarrow A=-B = -i/2 \\ A-B = -i \Leftrightarrow B = i/2 \end{cases}$$

$$* = \int_{\gamma} \frac{i \sin z \, dz}{2(z+i)} - \int_{\gamma} \frac{i \sin z \, dz}{2(z-i)}$$

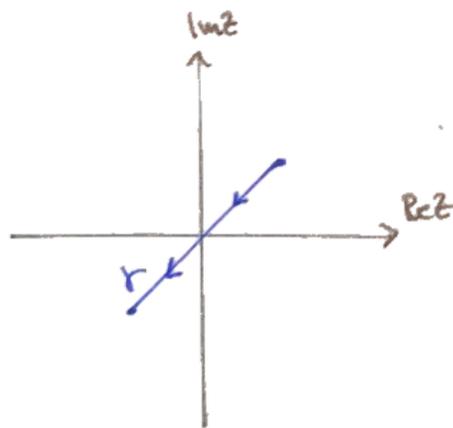
$$\left[f(z) = \frac{i \sin z}{2}, f(i) = \frac{i \sin i}{2}, f(-i) = -\frac{i \sin i}{2} \right]$$

$$2\pi i \left(-\frac{i \sin i}{2} - \frac{i \sin i}{2} \right) = -2\pi i^2 \sin i = -2\pi i^2 \frac{(e^{i^2} - e^{-i^2})}{2i} =$$

$$-\pi i (e^{-1} - e) = \pi i (e - e^{-1})$$

3.13

$$\int_{\gamma} \frac{z dz}{z^2 + 4} = \left[\begin{array}{l} u = z^2 + 4 \\ du = 2z dz \end{array} \right] =$$



$$\frac{1}{2} \int_{\gamma} \frac{du}{u} = \left[\frac{\log u}{2} \right]_{\gamma} = \left[\frac{\log(z^2 + 4)}{2} \right]_{\gamma}$$

Fixera log till principal, fungerar ty $(\pm 2 \pm 2i)^2 + 4 \notin \{-\pi, 0\}$
 π reella axeln

$$\int_{\gamma} \frac{z dz}{z^2 + 4} = \frac{1}{2} \left(\log((2+2i)^2 + 4) - \log((-2-2i)^2 + 4) \right) =$$

$$\frac{1}{2} \left(\log(4+8i) - \log(4+8i) \right) = 0$$

3.14 a) Nej b) Ja

$$3.15 \text{ a) } \int_{|z|=1} \frac{dz}{z(z-z)} , f(z) = \frac{1}{z-z} , f(0) = 1/z$$

$$\int_{|z|=1} \frac{dz}{z(z-z)} = 2\pi i \cdot f(0) = \pi i$$

b) $f'(z) = \frac{1}{z(z-z)}$, Saknar primitiv funktion ty

integralen i a) inte gar 0, for inte omsluta origo

3.16 Nej, motexempel: $f(z) = z^2$

$\gamma: 0 \rightarrow i$, $z(t) = it$, $0 \leq t \leq 1$, $dz = i dt$

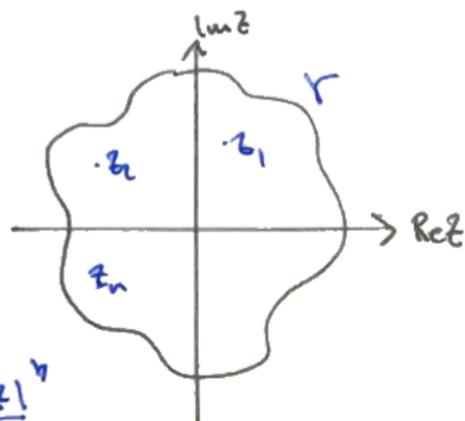
$$\int_{\gamma} f(z) dz = \int_0^1 z(t) \cdot i dt = \int_0^1 -zt dt = \left[-t^2 \right]_0^1 = -1$$

$\operatorname{Re} z(t) = 0 \Rightarrow \int_{\gamma} \operatorname{Re} f(z) = 0$, dock sant for reella tal

3.19

Enligt korollarium 3.21 kan γ deformeras hur som helst sa lange ingen singularitet passeras, darav lat $\gamma: |z| = R$

$$(z-z_1)(z-z_2) \dots (z-z_n) \geq \frac{1}{2} |z| \cdot |z|^n = \frac{|z|^{n+1}}{2}$$



$$\Rightarrow \left| \int_{\gamma} \frac{dz}{(z-z_1) \dots (z-z_n)} \right| \leq \int_{\gamma} \frac{2 dz}{|z|^{n+1}} = \int_0^{2\pi} \frac{2 dz}{R^{n+1}} = \frac{2}{R^{n+1}} \left[z \right]_0^{2\pi} = \frac{4\pi i}{R^{n+1}} \rightarrow 0 \text{ da}$$

$R \rightarrow \infty$

$$\text{Vielm. att } \int_{|z|=1} f(z) dz = 0$$

3.36 a)

$$f(z) = \frac{1}{z^2}$$

1. Entligt definitionen

$$z(t) = e^{it}, \quad dz = ie^{it} dt, \quad 0 \leq t \leq 2\pi$$

$$\int_{|z|=1} f(z) dz = \int_{|z|=1} \frac{dz}{z^2} = \int_0^{2\pi} \frac{ie^{it}}{e^{2it}} dt = \left[e^{-it} \right]_0^{2\pi} = e^{-2\pi i} - e^0 = 0$$

2. Cauchy's integralformel (deriverad), $g(z) = 1$

$$\int_{|z|=1} f(z) dz = \int_{|z|=1} \frac{g(z)}{(z-0)^2} dz = i2\pi \cdot g'(0) = 0$$

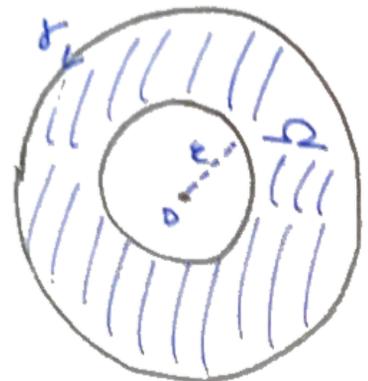
3. Primitiv funktion, endast $\mathbb{C} \setminus \{0\}$

$$\int_{|z|=1} f(z) dz = \int_{|z|=1} \frac{dz}{z^2} = \left[-\frac{1}{z} \right]_1^1 = 0$$

4. Cauchy's integralsats (med h_0^1)

$$0 = \int_{\partial\Omega} \frac{dz}{z^2} = \int_{\gamma} \frac{dz}{z^2} - \int_{\partial D_r} \frac{dz}{z^2} = \int_{\gamma} \frac{dz}{z^2}$$

$\underbrace{\int_{\partial D_r} \frac{dz}{z^2}}_{=0 \text{ (per def)}}$



5. Entligt kollarium 3.21 kan $\gamma: |z|=1$ deformation till $\hat{\gamma}: |z|=R, R>1$

$$\left| \int_{\hat{\gamma}} \frac{dz}{z^2} \right| \leq \int_{\hat{\gamma}} \frac{|dz|}{|z|^2} \leq \frac{2\pi R}{R^2} = \frac{2\pi}{R} \rightarrow 0 \text{ di } R \rightarrow \infty$$

$$f(z) = \frac{1}{\sin^2 z}, \quad z = e^{i\theta}, \quad dz = ie^{i\theta} d\theta, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\begin{aligned}
 \text{b) } \int_{|z|=1} \frac{dz}{\sin^2 z} &= \int_{|z|=1} \frac{-4dz}{e^{2iz} - 2 + e^{-2iz}} = -4i \int_0^{2\pi} \frac{e^{i\theta} d\theta}{e^{2ie^{i\theta}} - 2 + e^{-2ie^{i\theta}}} = \\
 -4i \left(\int_0^{\pi} \frac{e^{i\theta} d\theta}{C} + \int_{\pi}^{2\pi} \frac{e^{i\theta} d\theta}{C} \right) &= \left[\begin{array}{l} t = \theta - \pi, \pi \rightarrow 0 \\ dt = d\theta, 2\pi \rightarrow \pi \end{array} \right] = -4i \left(\int_0^{\pi} \frac{e^{i\theta} d\theta}{C} + \int_0^{\pi} \frac{e^{it} dt}{e^{2ie^{it}} - 2 - 2ie^{it}} \right) \\
 &= -4i \left(\int_0^{\pi} \frac{e^{i\theta} d\theta}{C} - \int_0^{\pi} \frac{e^{i\theta} d\theta}{C} \right) = 0, \quad \text{by } e^{i\pi} = -1
 \end{aligned}$$

$$\text{c) } f(z) = \frac{e^{iz^2}}{z^2 + 4} = \frac{e^{iz^2}}{(z-2i)(z+2i)}$$

$$\int_{|z|=1} f(z) dz = 0 \quad (\text{Cauchy's integrals})$$

$$\text{d) } f(z) = (\operatorname{Im} z)^2, \quad z = e^{it} = \cos t + i \sin t$$

$$f(z) = \sin^2 t, \quad 0 \leq t \leq 2\pi, \quad dz = 2i \sin t \cos t dt$$

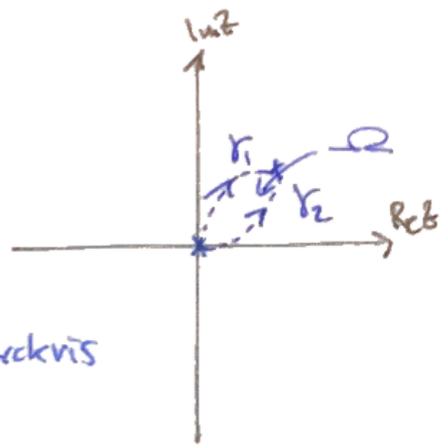
$$\int_{|z|=1} f(z) dz = \int_0^{2\pi} 2 \sin^3 t \cos t dt = \left[\begin{array}{l} u = \sin t, \quad 0 \rightarrow 0 \\ du = \cos t dt \end{array} \right] =$$

$$\int_0^0 2u^3 dt = 0$$

$$\text{e) } f(z) = z|z|^4$$

$$\int_{|z|=1} z|z|^4 dz = \int_{|z|=1} z dz = 0 \quad (\text{Cauchy's integrals})$$

3.3b a) Korollarium 3.20



$$\text{om } \int_{\gamma_1} \bar{z} dz = \int_{\gamma_2} \bar{z} dz \quad \text{och } \gamma_1 \neq \gamma_2 \text{ är sträckvis}$$

slöta kurvor med samma start- & slutpunkter, måste f vara holomorf.

Det stämmer inte för \bar{z} inte är holomorf på \mathbb{C} (inte ens på hela \mathbb{C}).

$$b) \gamma_1: z(t) = (1+i)t \Rightarrow dz = 1+i, \quad 0 \leq t \leq 1$$

$$\int_{\gamma_1} \bar{z} dz = \int_0^1 (1+i)t dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2}$$

$$\gamma_2: z(t) = t + it^2, \quad dz = 1 + 2it, \quad 0 \leq t \leq 1$$

$$\int_{\gamma_2} \bar{z} dz = \int_0^1 (t - it^2)(1 + 2it) dt = \int_0^1 (t + 2it^2 - it^2 + 2t^3) dt =$$

$$\int_0^1 (2t^3 + it^2 + t) dt = \left[\frac{2t^4}{4} + \frac{it^3}{3} + \frac{t^2}{2} \right]_0^1 = 1 + \frac{i}{3} \neq \frac{1}{2}$$

3.18 Anta att $|a| < 1$

$$\int_0^{2\pi} \ln|1 - ae^{i\theta}| d\theta = \left[z = e^{i\theta}, dz = ie^{i\theta} d\theta = iz d\theta \right] =$$

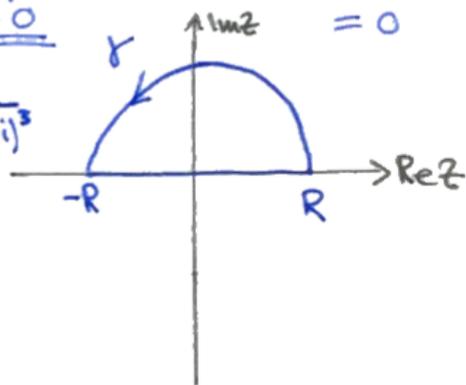
$$\int_{|z|=1} \frac{\ln|1 - az|}{iz} dz; \text{ undersök istället integralen } \int_{|z|=1} \frac{\log(1 - az)}{iz} dz =$$

$$\int_{|z|=1} \frac{-i \log(1 - az)}{z - 0} dz = 2\pi \log(1) = 0, \text{ principalgrenen g\u00e5r bra ty } 1 - az > 0$$

$$\int_{|z|=1} \frac{\log(1 - az)}{iz} dz = \int_0^{2\pi} \log(1 - ae^{i\theta}) d\theta = \underbrace{\int_0^{2\pi} \ln|1 - ae^{i\theta}| d\theta}_{=0} + i \underbrace{\int_0^{2\pi} \arg(1 - ae^{i\theta}) d\theta}_{=0} = 0$$

3.20

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}, f(z) = \frac{1}{(z+i)^2} \Rightarrow f'(z) = \frac{-2}{(z+i)^3}$$



$$\int_{\gamma} \frac{dz}{(1+z^2)^2} = \int_{-R}^R \frac{dz}{(1+z^2)^2} + \int_{C_R^+} \frac{dz}{(1+z^2)^2} =$$

$$\int_{\gamma} \frac{dz}{(z+i)^2 (z-i)^2} = 2\pi i \cdot f'(i) = 2\pi i \cdot \frac{-2}{-8i} = \pi/2$$

$$\left| \int_{C_R^+} \frac{dz}{(1+z^2)^2} \right| \leq \int_{C_R^+} \frac{|dz|}{|1+z^2|^2} \leq \int_{C_R^+} \frac{|dz|}{2|z|^4} \leq \frac{\pi R}{2R^4} = \frac{\pi}{2R^3} \rightarrow 0 \text{ d\u00e5 } R \rightarrow \infty$$

$$\int_{-R}^R \frac{dz}{(1+z^2)^2} = \int_{\gamma} \frac{dz}{(1+z^2)^2} - \int_{\gamma_R^+} \frac{dz}{(1+z^2)^2} \rightarrow \int_{-a}^a \frac{dx}{(1+x^2)^2} = \pi/2 \quad \text{da } R \rightarrow \infty$$

3.30 $\alpha \in \mathbb{C}, |\alpha| \neq 1$

$$\int_0^{2\pi} \frac{d\theta}{1-2\alpha \cos\theta + \alpha^2}, \quad \text{undersök integralen} \quad \int_{|z|=1} \frac{dz}{(z-\alpha)(z-1/\alpha)} =$$

$$\left[\begin{array}{l} z = e^{i\theta} \\ dz = ie^{i\theta} \end{array} \right] = \int_0^{2\pi} \frac{ie^{i\theta} d\theta}{(e^{i\theta}-\alpha)(e^{i\theta}-1/\alpha)} = \int_0^{2\pi} \frac{ie^{i\theta} d\theta}{e^{2i\theta} - \frac{e^{i\theta}}{\alpha} - \alpha e^{i\theta} + 1} =$$

$$\int_0^{2\pi} \frac{id\theta}{\alpha e^{i\theta} - 1 - \alpha^2 + \alpha e^{-i\theta}} = -i\alpha \int_0^{2\pi} \frac{d\theta}{1 - \alpha(e^{i\theta} + e^{-i\theta}) + \alpha^2} = -i\alpha \int_0^{2\pi} \frac{d\theta}{1 - 2\alpha \cos\theta + \alpha^2}$$

$$\int_0^{2\pi} \frac{d\theta}{1-2\alpha \cos\theta + \alpha^2} = \frac{-1}{i\alpha} \int_{|z|=1} \frac{dz}{(z-\alpha)(z-1/\alpha)} = \frac{i}{\alpha} \int_{|z|=1} \frac{dz}{(z-\alpha)(z-1/\alpha)}$$

① $|\alpha| < 1, f(z) = \frac{1}{z-1/\alpha}$

$$\frac{i}{\alpha} \int_{|z|=1} \frac{dz}{(z-\alpha)(z-1/\alpha)} = \frac{2\pi i^2}{\alpha} \cdot f(\alpha) = \frac{-2\pi}{\alpha} \cdot \frac{1}{\alpha-1/\alpha} = \frac{2\pi}{1-\alpha^2}$$

② $|\alpha| > 1, f(z) = \frac{1}{z-\alpha}$

$$\frac{i}{\alpha} \int_{|z|=1} \frac{dz}{(z-\alpha)(z-1/\alpha)} = \frac{2\pi i^2}{\alpha} \cdot f(1/\alpha) = \frac{-2\pi}{\alpha} \cdot \frac{1}{1/\alpha-\alpha} = \frac{2\pi}{\alpha^2-1}$$