

## Kapitel 2

21  $z^2 = 8 - 6i$ ,  $z = x + iy$

$$(x+iy)^2 = 8 - 6i \Leftrightarrow x^2 + 2xyi - y^2 = 8 - 6i \Rightarrow$$

$$\begin{cases} x^2 - y^2 = 8 \\ 2xy = -6 \Leftrightarrow 2x^2 = 18 \Rightarrow x = \pm 3 \\ x^2 + y^2 = 10 \qquad \qquad 2xy = -6 \Rightarrow y = \pm 1 \end{cases}$$

$$z_1 = 3 - i, z_2 = -3 + i$$

22

$$R(z) = \frac{z^2 + 2z + 5}{z^3 - z^2 + z - 1}$$

Nullstellen:  $z^2 + 2z + 5 = (z+1)^2 + 4 = (z+1)^2 - 4i^2 = (z+1 - 2i)(z+1 + 2i) = (z - (-1 + 2i))(z - (-1 - 2i))$

$$z_1 = -1 + 2i, z_2 = -1 - 2i$$

Pole:  $z^3 - z^2 + z - 1$ ,  $z = 1$  är ein Pol

$$\begin{array}{r} \overline{z^2 + 1} \\ \hline z^3 - z^2 + z - 1 \end{array} \left| \begin{array}{l} z-1 \\ -(z^3 - z^2) \end{array} \right.$$

$$z-1$$

$$\begin{array}{r} -(z-1) \\ 0 \end{array}$$

$$z^2 = -1 = i^2 \Leftrightarrow z = \pm i \text{ sind ordn. Pole}$$

$$2.3 \quad z^2 - 4z = 4 - 2i \Leftrightarrow (z - 2i)^2 + 4 = 4 - 2i \Leftrightarrow \\ (z - 2i)^2 = -2i \Leftrightarrow w^2 = -2i, \quad w = z - 2i$$

$$w = x + iy \Rightarrow x^2 - 2ixy - y^2 = -2i \Rightarrow$$

$$\begin{cases} x^2 - y^2 = 0 \\ 2xy = -2 \\ x^2 + y^2 = 2 \end{cases} \Rightarrow \begin{cases} 2x^2 = 2 \Rightarrow x = \pm 1 \\ 2xy = 2 \Rightarrow y = \pm 1 \end{cases}$$

$$w = \pm(1-i) \Rightarrow z_1 = 1+i, z_2 = -1+3i$$

$$2.4 \quad z^4 = 1+i$$

$$z = re^{iw} \Rightarrow z^4 = r^4 e^{4iw}$$

$$1+i = \sqrt{2}e^{i\pi/4} \Rightarrow \begin{cases} r^4 = \sqrt{2} \Leftrightarrow r = 2^{1/8} \\ 8t/4 + 2\pi k = 4w \Leftrightarrow w = 8t/16 + \frac{\pi}{2}k, \quad k=0,1,2,3 \end{cases}$$

$$2.5 \quad z^6 = (2-1)^6 \Leftrightarrow \left(\frac{2-1}{2}\right)^6 = 1 \Leftrightarrow w^6 = 1 \quad (w = 1 - \frac{1}{2}i)$$

$$w = re^{it} \Rightarrow w^6 = r^6 e^{-6it} \quad \left\{ \Rightarrow \begin{cases} r = 1 \\ 0 + 2\pi k = 6t \Rightarrow t = \frac{2\pi}{3}k \quad k=0,1,2,3,4,5 \end{cases} \right.$$

$$w = e^{i\pi/3}, \quad k=0 \text{ sakuur 10minus}$$

$$w_1 = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad w_5 = \frac{1}{2} - i\frac{\sqrt{3}}{2}, \quad w_3 = -1$$

$$z_1 = \frac{1}{\frac{1}{2} - i\frac{\sqrt{3}}{2}} = \frac{\frac{1}{2} + i\frac{\sqrt{3}}{2}}{\frac{1}{4} + \frac{3}{4}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad z_5 = \frac{1}{2} - i\frac{\sqrt{3}}{2}, \quad z_3 = -\frac{1}{2}$$

$$w_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow z_2 = \frac{1}{\frac{3}{2} - i\frac{\sqrt{3}}{2}} = \frac{2}{3 - i\sqrt{3}} = \frac{2(3 + i\sqrt{3})}{12} = \frac{1}{2} + i\frac{\sqrt{3}}{6}$$

$$z_4 = \frac{1}{2} - i\frac{\sqrt{3}}{6}$$

$$z_{1,5} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \quad z_{2,4} = \frac{1}{2} \pm i\frac{\sqrt{3}}{6}, \quad z_3 = -\frac{1}{2}$$

$$2.6 \quad \bar{z} = z^n, \quad n \in \mathbb{Z}^+$$

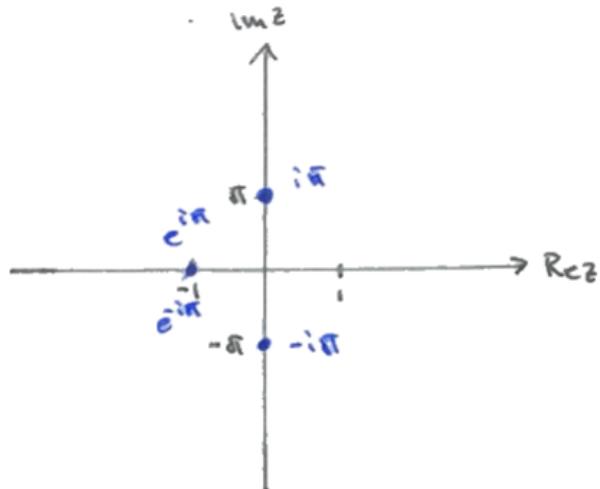
$$\bar{z}z = z^{n+1} \Leftrightarrow |z|^2 = z^{n+1}$$

$$z = re^{iw} \Rightarrow |z| = r, \quad z^{n+1} = r^{n+1} e^{i w(n+1)}$$

$$\left. \begin{array}{l} |(z)| = |z|^2 = r^2 \\ |z^{n+1}| = |z|^{n+1} = r^{n+1} \\ \arg |z|^2 = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} r^{n+1} = r^2 \Rightarrow r=1 \\ 2\pi k = w(n+1) \Rightarrow w = \frac{2\pi k}{n+1}, \quad k=0, \dots, n \end{array} \right.$$

$$z_1 = 0, \quad z_k = e^{\frac{2\pi ik}{n+1}}, \quad k=0, \dots, n+1$$

2.7



$$2.8 \quad a) \quad e^{1+i} = e \cdot e^i = e(\cos 1 + i \sin 1)$$

$$b) \quad |e^{-3+2i}| = e^{-3} |\cos 2 + i \sin 2| = e^{-3} \sqrt{\cos^2 2 + \sin^2 2} = e^{-3}$$

$$c) \quad \frac{e^{2-i}}{e^{-1+i}} = e^{2-i - (-1+i)} = e^{3-2i} = e^3 (\cos 2 - i \sin 2)$$

$$d) \quad \cos(1+i) = \frac{e^{i(1+i)} + e^{-i(1+i)}}{2} = \frac{(e^{-1+i} + e^{1-i})}{2} =$$

$$\begin{aligned} & (e^{-1}(\cos 1 + i \sin 1) + e(\cos 1 - i \sin 1))/2 = (\cos 1(e^{-1} + e)) + i \sin 1(e^{-1} - e) / 2 \\ & = \underline{(e + e^{-1})} (\cos 1 + i \sin 1) \end{aligned}$$

$$e) \sin(\delta/2 - i\ln a) = (e^{i(\delta/2 - i\ln a)} - e^{-i(\delta/2 - i\ln a)})/2i = \\ (e^{i\ln a + i\delta/2} - e^{-i\ln a - i\delta/2})/2i = (a(\cos\delta/2 + i\sin\delta/2) - \bar{a}^i(\cos\delta/2 - i\sin\delta/2))$$

$$= (ai - \bar{a}^i)/2i = (a + \bar{a}^i)/2$$

$$f) |t_{\text{anial}}| = \left| \frac{1}{i} \frac{e^{ia} - e^{-ia}}{e^{it} + e^{-it}} \right| = \left| \frac{1}{i} \frac{e^a - e^{-a}}{e^{ia} + e^{-ia}} \right| = \left| \frac{1}{i} \frac{1 - e^{2a}}{1 + e^{2a}} \right| = \\ \left| \frac{1 - e^{2a}}{1 + e^{2a}} \right| = \frac{|1 - e^{2a}|}{|1 + e^{2a}|} = \frac{e^{2a} - 1}{e^{2a} + 1}$$

$$2.9 \quad \left| \frac{1}{e^{i\omega t} - 1} \right| = \frac{1}{|\cos\omega t + i\sin\omega t - 1|} = 1/\sqrt{|\cos^2\omega t - \sin^2\omega t + i(2\sin\omega t \cos\omega t - \cos^2\omega t - \sin^2\omega t)|} =$$

$$1/\sqrt{|\sin\omega t(2\cos\omega t - 1)|} = 1/\sqrt{|\cos^2\omega t - \sin^2\omega t + 2i\sin\omega t \cos\omega t - \cos^2\omega t - \sin^2\omega t|} = 1/\sqrt{2|\sin\omega t \cos\omega t - \sin^2\omega t|} =$$

$$1/(2|\sin\omega t| |\cos\omega t - \sin\omega t|) = 1/(2|\sin\omega t| |\sin(\pi/2 - \omega t) - \cos(\pi/2 - \omega t)|)$$

$$= 1/(2|\sin\omega t| |\cos(\pi/2 - \omega t) - i\sin(\pi/2 - \omega t)|) = \frac{1}{2|\sin\omega t|}$$

$$2.10 \quad a) \quad e^{i\delta/5} = \cos\delta/5 + i\sin\delta/5 \Rightarrow 1:2 \text{ Quadranten}$$

$$b) \quad 2e^{3i\delta/4} \rightarrow 2:2$$

$$c) \quad e^{i\pi + i\delta/3} = e^{i\pi/3 + i\pi} = e^{5i\pi/3} \Rightarrow 4:2$$

$$d) \quad e^{1+4i} = e \cdot e^{4i} = e(\cos 4 + i\sin 4) \Rightarrow 3:2$$

$$z = a + bi$$

$$2.11 \quad e^{iz} = 3i \Leftrightarrow e^{ia} e^{ib} = 3e^{i\pi/2} \Leftrightarrow$$

$$\begin{cases} e^{ia} = 3 \\ 2b = \pi/2 + 2\pi k, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} a = \ln(3)/2 \\ b = \pi/4 + \pi k, k \in \mathbb{Z} \end{cases} \Rightarrow$$

$$z = \frac{\ln 3}{2} + i\left(\frac{\pi}{4} + \pi k\right), k \in \mathbb{Z}$$

$$2.16 \quad \sin z = 0 \Leftrightarrow \frac{e^{iz} - e^{-iz}}{2i} = 0 \Leftrightarrow e^{iz} = e^{-iz} \Leftrightarrow$$

$$[w = e^{iz}] \quad w = \sqrt{w} \Leftrightarrow w^2 = 1 \Leftrightarrow w = \pm 1 \Rightarrow$$

$$\textcircled{1} \quad e^{iz} = 1 \Leftrightarrow iz = \log(1) \Leftrightarrow iz = \ln 1 + i(0 + 2\pi k) \Leftrightarrow$$

$$z_1 = 2\pi k, k \in \mathbb{Z}$$

$$\textcircled{2} \quad e^{iz} = -1 \Leftrightarrow iz = \log(-1) \Leftrightarrow iz = \ln(-1) + i(\pi + 2\pi k) \Leftrightarrow$$

$$z_2 = \pi + 2\pi k \Rightarrow z = \pi k, k \in \mathbb{Z}$$

$$\cos z = 0 \Leftrightarrow \frac{e^{iz} + e^{-iz}}{2} = 0 \Leftrightarrow e^{iz} = -e^{-iz} \Leftrightarrow [w = e^{iz}]$$

$$w = -\sqrt{w} \Leftrightarrow w^2 = -1 \Leftrightarrow w = \pm i \Rightarrow$$

$$\textcircled{1} \quad e^{iz} = i \Leftrightarrow iz = \ln(|i| + i(\frac{\pi}{2} + 2\pi k)) \Leftrightarrow z_1 = \frac{\pi}{2} + 2\pi k \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow$$

$$\textcircled{2} \quad e^{iz} = -i \Leftrightarrow iz = \ln(|-i| + i(\frac{3\pi}{2} + 2\pi k)) \Leftrightarrow z_2 = \frac{3\pi}{2} + 2\pi k \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow$$

$$z = \pi/2 + \pi k$$

$$2.18 \quad \tan z = 3i \Leftrightarrow \frac{1}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = 3i \Leftrightarrow [w = e^{iz}]$$

$$\frac{w - w^{-1}}{w + w^{-1}} = -3 \Leftrightarrow \frac{w^2 - 1}{w^2 + 1} = -3 \Leftrightarrow w^2 - 1 = -3w^2 - 3 \Leftrightarrow$$

$$4w^2 = -2 \Leftrightarrow w^2 = -\frac{1}{2} \Leftrightarrow w = \pm i/\sqrt{2}$$

$$\textcircled{1} \quad e^{iz} = i/\sqrt{2} \Leftrightarrow iz = \ln\left(\frac{1}{\sqrt{2}}\right) + i\left(\frac{\pi}{2} + 2\pi k\right) \Leftrightarrow$$

$$z_1 = \frac{\pi i}{2} + 2\pi k + i \ln \sqrt{2} \Leftrightarrow z_1 = \frac{\pi i}{2} + 2\pi k + i \frac{\ln 2}{2}$$

$$\textcircled{2} \quad e^{iz} = -i/\sqrt{2} \Leftrightarrow iz = \ln\left|\frac{-i}{\sqrt{2}}\right| + i\left(\frac{3\pi}{2} + 2\pi k\right) \Leftrightarrow$$

$$z_1 = \frac{3\pi i}{2} + 2\pi k + i \frac{\ln 2}{2}$$

$$z = \frac{\pi i + i \ln 2}{2} + 2\pi k, \quad k \in \mathbb{Z}$$

$$220 \quad \text{a)} \quad \log(1+i) = \frac{\ln 2}{2} + i\left(\frac{\pi}{4} + 2\pi k\right), \quad k \in \mathbb{Z}$$

$$\text{b)} \quad \text{Log}(1+i) = \frac{\ln 2}{2} + \frac{i\pi}{4} = \frac{2\ln 2 + i\pi}{4}$$

$$\text{c)} \quad \text{Log}_n(1+i) = \frac{\ln 2}{2} + \frac{i\pi}{4}$$

$$\text{d)} \quad \log(-i) = \ln|-i| + i(3\pi/2 + 2\pi k) = i(3\pi/2 + 2\pi k)$$

$$\text{e)} \quad \text{Log}(-i) = -i\pi/2$$

$$\text{f)} \quad \text{Log}_n(-i) = 3\pi i/2$$

$$g) e^{\log(2+5i)} = 2+5i$$

$$h) e^{\operatorname{Log}(2+5i)} = 2+5i$$

$$i) \operatorname{Log}(e^{2+5i}) = \operatorname{Log}(e^2 e^{5i}) = 2 + \operatorname{Log}(e^{5i}) =$$

$$2 + \operatorname{Log}(\cos 5 + i \sin 5) = 2 + \ln 1 + i(5 - 2\pi) = 2 + i(5 - 2\pi)$$

$$j) \operatorname{Log}_n(e^{2+5i}) = 2 + \operatorname{Log}_n(\cos 5 + i \sin 5) = 2 + 5i$$

$$2.22 \text{ a) } i^3 = e^{3\operatorname{Log}(i)} = e^{3(\ln 1 + i(\pi/2 + 2\pi k))} = e^{3i(\pi/2 + 2\pi k)} = -1$$

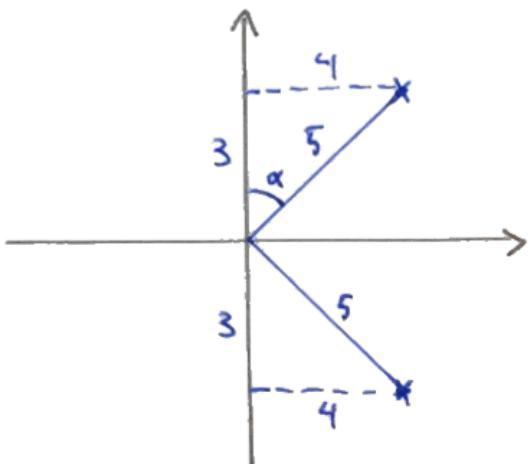
$$\text{b) } i\sqrt{3} = e^{i/3(\pi/2 + 2\pi k)} = \begin{cases} \cos(\pi/6) + i \sin(\pi/6) = \sqrt{3}/2 + i/2 \\ \cos(5\pi/6) + i \sin(5\pi/6) = -\sqrt{3}/2 + i/2 \\ \cos(3\pi/2) + i \sin(3\pi/2) = -i \end{cases}$$

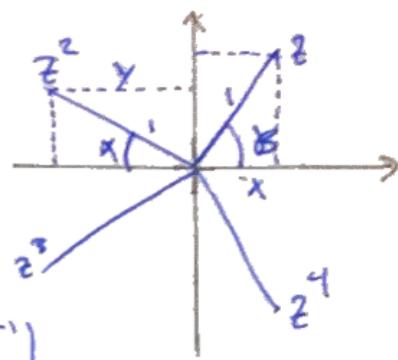
$$\text{c) } i^i = e^{i(\pi/2 + 2\pi k)} = e^{-i(\pi/2 + 2\pi k)}$$

$$\text{d) } i^\pi = e^{\pi i(\pi/2 + 2\pi k)}, \quad k \in \mathbb{Z}$$

$$2.13 |z| = 5, \operatorname{Arg}(z-4) = \pi/2$$

$$z = 3 + 4i$$





2.17 Visa att  $\cos\left(\frac{2\pi}{5}\right) = \frac{-1 + \sqrt{5}}{4}$

$$z = e^{2\pi i/5} \rightarrow x = \cos\left(\frac{2\pi}{5}\right) = \frac{1}{2}(z + z^{-1})$$

$$1 + z + z^2 + z^3 + z^4 = \frac{z^5 - 1}{z - 1} = \frac{e^{2\pi i} - 1}{z - 1} = 0 \Leftrightarrow$$

$$\textcircled{1} \quad z^{-2} + z^{-1} + 1 + z + z^2 = 0$$

$$z + z^{-1} = 2x$$

$$x^2 = \frac{1}{4}(z^2 + zz^{-1} + z^{-2}) = \frac{1}{4}(z^2 + z^{-2}) + \frac{1}{2} \Leftrightarrow$$

$$z^2 + z^{-2} = 4x^2 - 2$$

$$\textcircled{1} \quad 4x^2 - 2 + 2x + 1 = 0 \Leftrightarrow x^2 + \frac{x}{2} - \frac{1}{4} = 0 \Leftrightarrow$$

$$\left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{1}{4} = 0 \Leftrightarrow \left(x + \frac{1}{4}\right)^2 = \frac{5}{16} \Rightarrow x = \pm \frac{\sqrt{5}}{4} - \frac{1}{4}$$

$$\Rightarrow x = \cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$$

2.21  $f$  är holomorf,  $e^f$  är konstant

$$e^f = c, c \in \mathbb{C}$$

$f' \cdot e^f = 0 \Leftrightarrow f' = 0 \Rightarrow f$  är konstant på  $\mathbb{D}$

2.23  $w = \frac{1}{i} \log \left( \frac{1-i\bar{z}}{1+i\bar{z}} \right)$

$$\tan w = \frac{1}{i} \frac{e^{wi} - e^{-wi}}{e^{wi} + e^{-wi}}$$

$$e^{wi} = e^{\log \left( \frac{1-i\bar{z}}{1+i\bar{z}} \right)} = \frac{1-i\bar{z}}{1+i\bar{z}}, e^{-wi} = \frac{1+i\bar{z}}{1-i\bar{z}}$$

$$\tan w = \frac{\frac{1-i\bar{z}}{1+i\bar{z}} - \frac{1+i\bar{z}}{1-i\bar{z}}}{\frac{1-i\bar{z}}{1+i\bar{z}} + \frac{1+i\bar{z}}{1-i\bar{z}}} = \frac{1}{i} \frac{(1-i\bar{z})^2 - (1+i\bar{z})^2}{(1-i\bar{z})^2 + (1+i\bar{z})^2} =$$

$$\frac{1}{i} \frac{1-2i\bar{z}-\bar{z}^2 - (1+2i\bar{z}-\bar{z}^2)}{1-2i\bar{z}-\bar{z}^2 + (1+2i\bar{z}-\bar{z}^2)} = \frac{-4\bar{z}}{2-2\bar{z}^2} = \frac{2z}{z^2-1}, z \neq \pm 1$$

Oberoende av gren

2.24  $(z^{\nu_2})$  beteckar principalgrenen av den komplexa kvadratrotten

a)  $f(z) = (z+1)^{1/2} \Rightarrow f'(z) = \frac{1}{2(z+1)^{1/2}}$

$$(z+1)^{1/2} = e^{\frac{\log(z+1)}{2}} = e^{\ln|z+1| + i\arg(z+1)}$$

Om principal  $\mathbb{C} \setminus \{-\infty, -i\}$  på den reella axeln

Om naturlig  $\mathbb{C} \setminus \{-1, \infty\}$  på den reella axeln

b)  $f(z) = (1+z^2)^{1/2}$

$$(1+z^2)^{1/2} = e^{\frac{\log(1+z^2)}{2}} = e^{\frac{1}{2}(\ln|1+z^2| + i\arg(1+z^2))}$$

Principal: Ej holomorf då  $|z| \geq 1$  på den imaginära axeln

Naturlig: Ej holomorf då  $|z| \leq 1$  på den imaginära axeln, samt båda reella axeln

c)  $f(z) = z(1+\bar{z}^2)^{1/2}$

$$z(1+\bar{z}^2) = e^{\log z + \log(1+\bar{z}^2)} = e^{\ln|z| + i\arg z} \cdot e^{\ln|1+\bar{z}^2| + i\arg(1+\bar{z}^2)}$$

Principal: Ej holomorf då  $|z| \leq 1$  på den imaginära axeln

Naturlig: Ej holomorf då  $|z| \geq 1$  på den imaginära axeln, samt båda reella axeln

2.28  $p$  ist Polynom mit  $p(\alpha) = 0$ ,

$$P(z) = (z - \alpha) q(z) \Rightarrow q(z) = \frac{p(z)}{z - \alpha}, \deg q = \deg p - 1$$

$$P'(z) = q(z) + (z - \alpha) q'(z)$$

$$P'(\alpha) = q(\alpha)$$

$$\text{Dann } P'(\alpha) = 0 \Rightarrow q(\alpha) = 0 \Rightarrow q(z) = (z - \alpha) r(z)$$

$$\deg r = \deg q - 1 \Rightarrow P(z) = (z - \alpha)^2 r(z) \Rightarrow$$

$z = \alpha$  ist eine doppelte Nullstelle. Davor müste  $P'(z) \neq 0$