

1.1 a) $(-1+i)(2-i) = -2+i+2i-i^2 = -1+3i$

b) $(1+i)^2 = 1+2i+i^2 = 2i$

c) $\frac{3+2i}{1-i} = \frac{3+2i}{1-i} \cdot \frac{(1+i)}{(1+i)} = \frac{3+3i+2i+i^2}{2} = \frac{1+5i}{2}$

d) $|1+2i| = \sqrt{1^2+2^2} = \sqrt{5}$

e) $\overline{-3-2i} = -3+2i$

f) $(1+i)\overline{(1+i)} = (1+i)(1-i) = 2$

1.2 a) $3i = 3e^{i\pi/2}$

b) $-1-i = \sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right) = \sqrt{2}e^{i5\pi/4}$

c) $\sqrt{3}+i = 2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) = 2e^{i\pi/6}$

1.3 a) $|1-i| = \sqrt{2}$

b) $|2+3i| = \sqrt{13}$

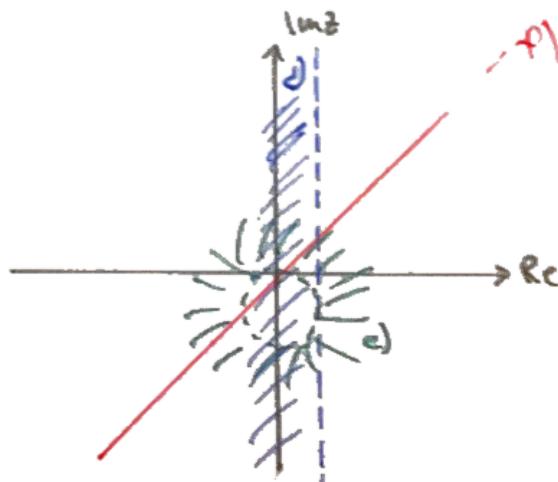
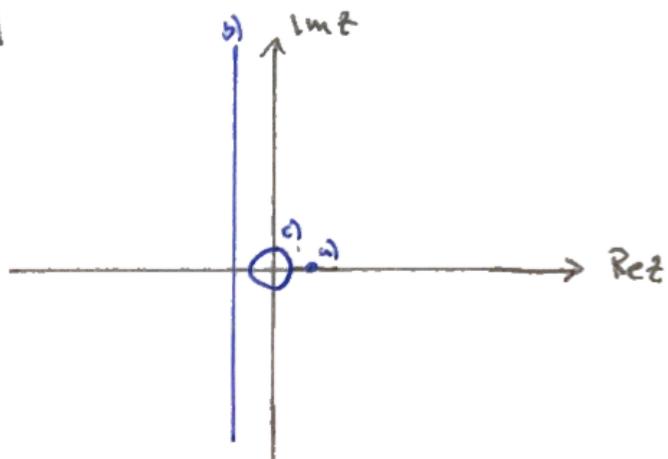
c) $|1-i+2+3i| = |3+2i| = \sqrt{13}$

d) $|(1-i)(2+3i)| = |(1-i)(2-3i)| = |2-3i-2i+3i^2| = |-1-5i| = \sqrt{26}$

e) $\left|\frac{1}{1-i}\right| = \left|\frac{1+i}{2}\right| = \frac{1}{2}|1+i| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

f) $\left|\frac{(1-i)(2+3i)^2}{(1-i)}\right| = |(2+3i)|^2 = 13$

1.4



1.5 a) Für z p: positive reellteileseiten (ink 0)

b) Für z p: positive imaginärteileseiten (ink 0)

c) $z=0$

1.6 a) 4 b) 2 c) 1/2 d) $(1+i)^3 = (\sqrt{2}e^{i\pi/4})^3 = \sqrt{2}^3 e^{i3\pi/4} \Rightarrow 2$

1.7 a)

$$f(z) = f(x+iy) = (2-i)(x+iy) = 2x + 2iy - ix - i^2y = \\ \underbrace{2x+y}_U + i(\underbrace{2y-x}_V), \text{ def } \subseteq \text{kont für alle } \mathbb{C}$$

b) $f(z) = (x+iy)^2 = \underbrace{x^2}_U + 2xyi + \underbrace{(iy)^2}_V = \underbrace{x^2-y^2}_U + i \cdot \underbrace{2xy}_V$, def \subseteq kont für alle \mathbb{C}

c) $f(z) = \overline{x+iy} = \underbrace{x}_U + i \underbrace{(-y)}_V, \text{ def } \subseteq \text{kont für alle } \mathbb{C}$

d) $f(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i \left(\frac{-y}{x^2+y^2} \right), \text{ def } \stackrel{\subseteq \text{ kont}}{\text{für alle}} \mathbb{C} \text{ außer origo}$

e) $f(z) = x+iy - (x-iy) = 2iy = i \cdot 2y, \text{ def } \subseteq \text{kont für alle } \mathbb{C}$

f) $f(z) = |x+iy-1| = \sqrt{(x-1)^2 + y^2} = \sqrt{\underbrace{x^2-2x+1}_U + y^2}, \text{ def } \subseteq \text{kont für alle } \mathbb{C}$

$$1.8 \text{ a)} \lim_{h \rightarrow 0} \frac{|h|^2}{h} = \lim_{h \rightarrow 0} \frac{h\bar{h}}{h} = \lim_{h \rightarrow 0} \bar{h} = 0$$

$$\text{b)} \lim_{h \rightarrow 0} \frac{\operatorname{Re} h}{h}$$

$$\begin{aligned} h = k &\Rightarrow \lim_{h \rightarrow 0} \frac{k}{h} = 1 \\ h = ik &\Rightarrow \lim_{h \rightarrow 0} \frac{0}{ik} = 0 \end{aligned} \quad \left. \begin{array}{l} \text{existerar intc} \\ \text{intc} \end{array} \right\} \Rightarrow$$

$$\text{c)} \lim_{h \rightarrow 0} \frac{(z+h)^3 - z^3}{h} = \lim_{h \rightarrow 0} \frac{z^3 + 3z^2h + 3zh^2 + h^3 - z^3}{h} = \underbrace{3z^2}_{h \rightarrow 0} + 3zh + h^2 = 3z^2$$

$$1.9 \text{ a)} f(z) = z^3 \Rightarrow f'(z) = 3z^2, \text{ overallt}$$

$$z = x+iy \Rightarrow z^3 = x^3 + i3x^2y + i^2 3xy^2 + (iy)^3 = \underbrace{x^3 - 3xy^2}_0 + i \underbrace{(3x^2y - y^3)}_V$$

$$v_x^1 = 3x^2 - 3y^2, v_y^1 = 3x^2 - 3y^2$$

$$v_y^1 = -6xy, -v_x^1 = -6xy$$

$$f'(z) = v_x^1 + iv_y^1 = 3(x^2 - 6xy - y^2) = 3z^2$$

Deriverbar \Leftrightarrow holomorf overallt

$$\text{b)} f(z) = \operatorname{Im} z, z = x+iy \Rightarrow \operatorname{Im} z = x$$

$$v_x^1 = 1, v_x^1 = 0 \Rightarrow \text{derivatan intc existerar, ej holomorf}$$

på nögot område

$$\text{c)} f(z) = 1/z \Rightarrow f'(z) = -1/z^2, z \neq 0$$

$$f(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i \left(\frac{-y}{x^2+y^2} \right)$$

$$v_x^1 = \frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, v_y^1 = \frac{-(x^2+y^2) + y(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_y^1 = \frac{-2x(x^2+y^2) - x(2y)}{(x^2+y^2)^2}, v_x^1 = \frac{-2x(x^2+y^2) + y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$f'(z) = u'_x + i v'_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} + i \left(\frac{2xy}{(x^2 + y^2)^2} \right) =$$

$$\frac{(y+xi)^2}{(y-xi)(y+xi)^2} = \frac{1}{(y-xi)^2} = \frac{1}{y^2 - 2xi - x^2} = \frac{1}{x^2 + 2xi - y^2} = \frac{-1}{z^2}$$

Deriverbar för \Leftrightarrow utan $z=0$, holomorf på hela området

d) $f(z) = |z|^2$

$$f'(z) = \begin{cases} 2z, & \text{för } z \geq 0 \\ -2z, & \text{för } z \leq 0 \end{cases}$$

$$f(z) = x^2 + y^2$$

$$\left. \begin{array}{l} u'_x = 2x, \quad u'_y = 0 \\ u'_y = 2y, \quad v'_x = 0 \end{array} \right\} \Rightarrow \text{intast deriverbar i } z=0, \quad f'(0)=0$$

g) holomorf på hela området

e) $f(x+iy) = e^y \cos x - ie^y \sin x$

$$u'_x = -e^y \sin x, \quad v'_y = -e^y \sin x$$

$$u'_y = e^y \cos x, \quad v'_x = -e^y \cos x \Rightarrow -v'_x = e^y \cos x$$

$$f'(x+iy) = -e^y (\cos x + i \sin x)$$

Deriverbar för alla z , holomorf på hela området (alla områden)

1.10 Ja!

$$\left. \begin{array}{l} p^{(1)}(0) = a_1 \\ p^{(2)}(0) = 2a_2 \\ p^{(3)}(0) = 6a_3 \end{array} \right\} \Rightarrow p^{(k)}(0) = k! a_k, \quad k \geq 4 \Rightarrow p^{(k)}(0) = 0$$

$$1.11 \quad v(x,y) = x^3 - y^3$$

Holomorf om $\left\{ \begin{array}{l} v_x^1 = v_y^1 \\ v_y^1 = -v_x^1 \end{array} \right.$

$$v_y^1 = -3y^2 = v_y^1 \Rightarrow v = -3xy^2 + \phi(y)$$

$$-v_x^1 = -3x^2 = v_y^1$$

$$v_y^1 = -6xy + \phi'(y) \neq -3x^2 \Rightarrow \text{Holomorf funktion funktions}$$

$$1.12 \quad v(x,y) = x^3 - 3xy^2$$

$$\left. \begin{array}{l} v_x^1 = 3x^2 - 3y^2 \Rightarrow v_{xx}^1 = 6x \\ v_y^1 = -6xy \Rightarrow v_{yy}^1 = -6x \end{array} \right\} \Rightarrow \Delta v = 0 \Rightarrow \text{Holomorf funktion}$$

$$f'(z) = f'(x+iy) = v_x^1 - i v_y^1 = 3x^2 - 3y^2 + i6xy$$

$$f'(z) = 3z^2 \Rightarrow f(z) = z^3 + K, K = iC$$

$$1.13 \quad v(x,y) = x - 4xy$$

$$\left. \begin{array}{l} v_x^1 = 1 - 4y \Rightarrow v_{xx}^1 = 0 \\ v_y^1 = -4x \Rightarrow v_{yy}^1 = 0 \end{array} \right\} \Rightarrow \Delta v = 0$$

$$f'(x+iy) = v_x^1 - i v_y^1 = 1 - 4y + 4x \cdot i$$

$$f'(z) = 4zi + 1 \Rightarrow f(z) = 2z^2i + z + iC$$

$$f(0) = i \Rightarrow iC = i \Rightarrow C = 1 \Rightarrow f(z) = 2z^2i + z + 1$$

$$1.14 \quad v(x,y) = x^2 + axy^2 + by$$

$$\left. \begin{array}{l} v_x^1 = 2x \Rightarrow v_{xx}^1 = 2 \\ v_y^1 = 2ay + b \Rightarrow v_{yy}^1 = 2a \end{array} \right\} a=1 \Rightarrow f \text{ holomorf}$$

$$f'(x+iy) = b - 2y + 2ix \Rightarrow f(z) = 2iz + b \Rightarrow f(z) = iz^2 + b \quad i, c \in \mathbb{C}$$

1.16 a) Nej! , exempelvis $u=v=x$ stämmer det inte på

b) $f=u+iw$, f är holomorf \Rightarrow

$$\left\{ \begin{array}{l} u_x^1 = v_y^1 \quad , \quad u_{xx}^1 + v_{yy}^1 = 0 \\ u_y^1 = -v_x^1 \quad , \quad v_{xx}^{11} + v_{yy}^{11} = 0 \end{array} \right.$$

$$(uv)_x^1 = u_x^1 v + u v_x^1 \Rightarrow (uv)_{xx}^{11} = u_{xx}^1 v + u_x^1 v_x^1 + u_x^1 v_x^1 + u v_{xx}^1$$

$$= u_{xx}^1 v + 2u_x^1 v_x^1 + u v_{xx}^{11}$$

$$(uv)_{yy}^{11} = u_{yy}^1 v + 2u_y^1 v_y^1 + u v_{yy}^{11}$$

$$(uv)_{xx}^{11} + (uv)_{yy}^{11} = v \underbrace{(u_{xx}^1 + v_{yy}^1)}_{=0} + 2(u_x^1 v_x^1 + u_y^1 v_y^1) + u \underbrace{(v_{xx}^{11} + v_{yy}^{11})}_{=0}$$

$$= 2(v_x^1 v_y^1 - u_x^1 u_y^1) = 0 \Rightarrow \text{homomorf}$$

1.18 $f = u + iv$ är holomorf utom i punkten $z=0$ \Leftrightarrow

$$\text{uppfyller } u + v = \frac{x+y}{x^2+y^2} \quad (x,y) \neq (0,0)$$

$$\begin{cases} u'_x = v'_y, & u''_{xx} + u''_{yy} = 0 \\ u'_y = -v'_x, & v''_{xx} + v''_{yy} = 0 \end{cases}$$

$$\textcircled{1} \quad u'_x + v'_x = \frac{x^2 + y^2 - (x+y) \cdot 2x}{(x^2+y^2)^2} = \underbrace{\frac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}}_C = u'_x - u'_y$$

$$\textcircled{2} \quad u'_y + v'_y = \underbrace{\frac{x^2 - y^2 - 2xy}{(x^2+y^2)^2}}_D$$

$$\textcircled{1} \quad u'_x - C + v'_y = D \Leftrightarrow 2u'_x = \frac{x^2 - y^2 - 2xy + y^2 - x^2 - 2xy}{(x^2+y^2)^2} \Leftrightarrow$$

$$2u'_x = \frac{-4xy}{(x^2+y^2)^2} \Leftrightarrow u'_x = \frac{-2xy}{(x^2+y^2)^2}$$

$$u'_y = u'_x - C = \frac{-2xy - y^2 + x^2 + 2xy}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$f' = u'_x - iu'_y = -\frac{2xy - i(x^2 - y^2)}{(x^2+y^2)} \Rightarrow$$

$$f'(z) = -\frac{iz^2}{z^2} = -\frac{i}{z^2} \Rightarrow f(z) = \frac{i}{z} + K, \quad K \in \mathbb{C}$$

$$f(z) = \frac{i}{z} + K = \frac{i\bar{z}}{|z|^2} + K = \frac{i(x-iy)}{x^2+y^2} + K = \frac{y+ix}{x^2+y^2} + K, \quad K = C_1 + C_2 i$$

$$, \quad C_1, C_2 \in \mathbb{R}$$

$$u = \operatorname{Re} f = \frac{y}{x^2+y^2} + C_1, \quad v = \operatorname{Im} f = \frac{x}{x^2+y^2} + C_2$$

$$u + v = \frac{x+y}{x^2+y^2} \Leftrightarrow C_1 + C_2 = 0 \Leftrightarrow C_2 = -C_1 \Rightarrow K = C(1-i)$$

$$f(z) = i/z + C(1-i)$$

1.19 a) $f = u + iv$ är holomorf

orthogonal om $\nabla u \cdot \nabla v = 0$

$$\left\{ \begin{array}{l} u'_x = v'_y, \quad u''_{xx} + u''_{yy} = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} u'_y = -v'_x, \quad v''_{xx} + v''_{yy} = 0 \end{array} \right.$$

$$\left. \begin{array}{l} \nabla u = (u'_x, u'_y) \\ \nabla v = (v'_x, v'_y) \end{array} \right\} \Rightarrow \nabla u \cdot \nabla v = u'_x v'_x + u'_y v'_y = 0$$

$$\|\nabla u\| = \sqrt{u'^2_x + u'^2_y} = \sqrt{v'^2_y + v'^2_x}$$

$$\|\nabla v\| = \sqrt{v'^2_x + v'^2_y}$$

$$f' = u'_x + iu'_y \Rightarrow |f'| = \sqrt{(u'_x - iu'_y)(u'_x + iu'_y)} = \sqrt{u'^2_x + u'^2_y}$$

b) $f'(p) \neq 0 \Leftrightarrow u(x, y) = \operatorname{Re} f(p); v(x, y) = \operatorname{Im} f(p) \Rightarrow$
 $f(p) = u(x, y) + iv(x, y)$

$$p = x + iy$$

$$\nabla u \perp u \Leftrightarrow \nabla v \perp v$$

$$\text{Eftersom } u \perp v \Rightarrow \nabla u \perp \nabla v$$

1.22 f vektormet p: hela \mathbb{C} (hel)

Vissa att $g(z) = \overline{f(\bar{z})}$ också är hel

$$f(z) = f(x+iy) = u(x,y) + i v(x,y)$$

$$\begin{cases} u'_x = v'_y \\ u'_y = -v'_x \end{cases} \quad \text{niobe varu uppflykt}$$

$$g(z) = g(x+iy) = \underbrace{u(x,-y)}_{a(x,y)} + i \cdot \underbrace{-v(x,-y)}_{b(x,y)}$$

$$\operatorname{Re} g = a, \operatorname{Im} g = b$$

$$a'_x = v'_x, a'_y = -v'_y, b'_x = -v'_x, b'_y = v'_y$$

$$\begin{aligned} u'_x &= v'_y \Rightarrow a'_x = b'_y \\ u'_y &= -v'_x \Rightarrow -a'_y = -(-b'_x) \Rightarrow a'_y = -b'_x \end{aligned} \quad \left. \right\} \text{stämmer}$$