

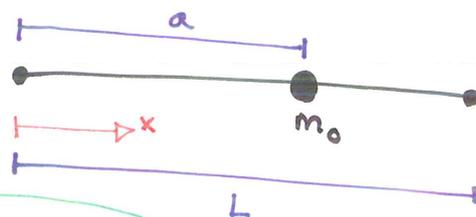
KAPITEL D

D.1 Formulera det matematiska problemet för små transversella svängningar.

$$\frac{\delta^2 U}{dt^2} - c^2 \frac{\delta^2 x}{\delta x^2} = -\frac{m_0 g}{\rho} \delta(x-a)$$

$$U(0, t) = U(L, t) = 0$$

$$U(x, t) = g(x), \quad U_t(x, 0) = h(x)$$



se FS
 $f = m_0 g \delta(x-a)$!

D.4 Bevisa 'förenklingsreglerna'!

$$a) \quad g(x) \delta'(x) = g(0) \delta'(x) - g'(0) \delta(x)$$

Vi vet att $g(x) \cdot \delta(x) = g(0) \delta(x)$

Derivering ger:

$$(g(x) \delta(x))' = g'(x) \delta(x) + g(x) \delta'(x) = \underline{g'(0) \delta(x)} + \underline{g(x) \delta'(x)}$$

$$(g(0) \delta(x))' = \underline{g(0) \delta'(x)}$$

$$\Rightarrow g(x) \delta'(x) = g(0) \delta'(x) - g'(0) \delta(x)$$

□

$$b) \boxed{g(x) \delta''(x) = g(0) \delta''(x) - 2g'(0) \delta'(x) + g''(0) \delta(x)}$$

$$\boxed{g(x) \cdot \delta''(x)} = g(x) (\delta'(x))' = (g(x) \delta'(x))' - g'(x) \delta'(x) =$$

$$= (g(0) \delta'(x) - g'(0) \delta(x))' - (g'(0) \delta'(x) - g''(0) \delta(x)) =$$

$$= g(0) \delta''(x) - g'(0) \delta'(x) - g'(0) \delta'(x) + g''(0) \delta(x) =$$

$$= \boxed{g(0) \delta''(x) - 2g'(0) \delta'(x) + g''(0) \delta(x)}$$

□

D.5

a) $e^x \delta(x-1) = e^1 \delta(x-1) = e \delta$

b) $x \delta' = 0 \cdot \delta' - \delta = -\delta$

c) $\cos(x) \delta' = \cos(0) \delta' + \sin(0) \delta = \delta'$

d) $x^2 \delta'' = 0 \cdot \delta'' - 2 \cdot 0 \cdot \delta' + 2 \cdot 1 \cdot \delta = 2\delta$

e) $\int g(t) \delta(t-1) dt = g(1) \int \delta(t-1) dt = g(1)$

D.6

Beräkna f' och f''

a) $f = e^{-x} \cdot \theta(x)$

$f' = -e^{-x} \theta(x) + e^{-x} \delta(x) = -e^{-x} \theta(x) + \delta(x)$

~~$f'' = e^{-x} \theta(x) - e^{-x} \delta(x) + \delta'(x)$~~ $f'' = e^{-x} \theta(x) - \delta(x) + \delta'(x)$

b) $f = e^{-|x|} = \begin{cases} e^x & , x < 0 \\ e^{-x} & , x > 0 \end{cases} = (1 - \theta(x)) e^x + e^{-x} \cdot \theta(x)$

$f' = -\delta(x) e^x + (1 - \theta(x)) e^x + -e^{-x} \theta(x) + e^{-x} \delta(x) = (1 - \theta(x)) e^x - e^{-x} \theta(x)$

$f'' = -\delta(x) e^x + (1 - \theta(x)) e^x + e^x \theta(x) - e^{-x} \delta(x) = (1 - \theta(x)) e^x + e^{-x} \theta(x) - 2 \cdot \delta(x)$

D.8 a) derivara $x^2 \Theta(x-1)$

$$f = x^2 \Theta(x-1)$$

$$f' = 2x \Theta(x-1) + 1 \cdot \delta(x-1)$$

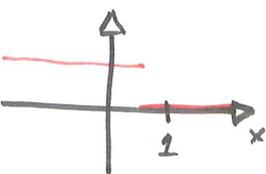
$$b) \int x \Theta(x-1) dx = \frac{1}{2} \int 2x \Theta(x-1) + \delta(x-1) - \delta(x-1) dx$$

$$= \frac{1}{2} (x^2 \Theta(x-1) - \Theta(x-1) + C) =$$

$$= \frac{1}{2} \Theta(x-1) (x^2 - 1) + C$$

~~$\frac{1}{2} x^2 \Theta(x-1)$~~

D.17 Skriv om följande distributioner mha Θ_a , δ_a & δ'_a .

a) $\Theta(-2x+1) =$  $=$ $\boxed{1 - \Theta_{1/2}}$

b) $\delta(2x) = \frac{1}{|2|} \delta(x) = \boxed{\frac{1}{2} \delta(x)}$

c) $\delta(-2x+1) = \boxed{\frac{1}{2} \delta_{1/2}}$

d) $\delta'(2x) =$

D.18 Beräkna följningarna

a) $\delta * e^{-x^2} = \boxed{e^{-x^2}}$

b) $\delta' * (e^{-x} \cdot \Theta(x)) = (\delta * (e^{-x} \cdot \Theta(x)))' = (e^{-x} \cdot \Theta(x))' = \boxed{\delta - e^{-x} \Theta(x)}$

c) $\delta_1 * \sin(2x) = \delta(x-1) \sin(2x) = \boxed{\sin(2x-2)}$

D.20

a) $(1 * \delta') * \Theta = 0 * \Theta = \boxed{0}$

b) $1 * (\delta' * \Theta) = 1 * \delta = \boxed{1}$

c) $\delta' * (1 * \Theta) =$ ej definierad eftersom

Associativa lagen gäller ej!

D.22

Utveckla $\delta'(x - \frac{\pi}{2})$ i en cosinusserie på intervallet $[0, \pi]$.

$$\delta'(x - \frac{\pi}{2}) = c_0 + \sum_{k=1}^{\infty} \alpha_k \cos\left(\frac{k\pi x}{\pi}\right)$$

$$c_0 = \frac{1}{\pi} \int_0^{\pi} \delta'(x - \frac{\pi}{2}) dx = \underline{\underline{0}}$$

$$\alpha_k = \frac{2}{\pi} \int_0^{\pi} \delta'(x - \frac{\pi}{2}) \cos(kx) dx =$$

$$= \frac{2}{\pi} \left(\left[\delta(x - \frac{\pi}{2}) \cos(kx) \right]_0^{\pi} + \int_0^{\pi} \delta(x - \frac{\pi}{2}) \sin(kx) \cdot k dx \right) =$$

↑ har värde 0
då $x = \pi, x = 0$.

Denna integral har
endast värde då
 $x = \pi/2$.

$$= \frac{2}{\pi} \left(0 + \delta(\pi/2 - \pi/2) \sin\left(\frac{\pi}{2}\right) \cdot k \right) = \underline{\underline{\frac{2k}{\pi} \sin\left(\frac{k\pi}{2}\right)}}$$

Detta ger:

$$\delta'(x - \pi/2) = \frac{2}{\pi} \sum_{k=1}^{\infty} k \sin\left(\frac{k\pi}{2}\right) \cdot \cos(kx)$$

D.23 Utveckla $\delta(r - \frac{1}{2})$ i ortogonalsystemet från övning 5.20

Från 5.20: $\lambda_k = k^2 \pi^2$, $\psi_k = \frac{\sin(k\pi r)}{k\pi r}$

$$\Rightarrow \delta(r - \frac{1}{2}) = \sum_{k=1}^{\infty} \frac{(\psi_k | \delta(r - \frac{1}{2}))}{(\psi_k | \psi_k)} \psi_k(r)$$

HALLÅ
 Vi använder det felaktiga ψ_k som står i facit till 5.20!
 $\psi_k(r) = \frac{\sin(k\pi r)}{r}$

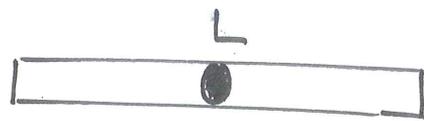
$$(\psi_k | \delta(r - \frac{1}{2})) = \int_0^1 \frac{\sin(k\pi r)}{k\pi r} \delta(r - \frac{1}{2}) \cdot r^2 dr = \frac{\sin(k\pi \frac{1}{2})}{2k\pi}$$

$$(\psi_k | \psi_k) = \int_0^1 \frac{\sin^2(k\pi r)}{k^2 \pi^2} \cdot r^2 dr = \dots = \frac{1}{2k^2} = \frac{1}{2}$$

$$\Rightarrow \delta(r - \frac{1}{2}) = \sum_{k=1}^{\infty} \sin(\frac{k\pi}{2}) \cdot \sin(\frac{k\pi r}{1}) \cdot \frac{1}{2}$$

↑ facit är fel i 5.20, här eller båda.

D.24 Lös diffusionsproblemet



$$\begin{cases} \text{PDE} & \left\{ \begin{array}{l} \frac{\partial U}{\partial t} - D \frac{\partial^2 U}{\partial x^2} = 0 \\ U_x'(0, t) = U_x'(L, t) = 0 \\ U(x, 0) = M \cdot \delta(x - \frac{L}{2}) \end{array} \right. \end{cases} \quad , \quad t > 0, \quad 0 < x < L$$

∵ har homogena Neumannvillkor \Rightarrow Vi sätter en cosinusserie!

$$U(x, t) = c_0 + \sum_{k=1}^{\infty} \alpha_k(t) \cdot \cos\left(\frac{k\pi}{L}x\right)$$

Insättning i PDE & termvis derivation ger:

$$\sum_{k=1}^{\infty} (\alpha_k'(t) + D \left(\frac{k\pi}{L}\right)^2 \alpha_k(t)) \cos\left(\frac{k\pi}{L}x\right) = 0$$

Entydighet ger:

$$\alpha_k' + D \left(\frac{k\pi}{L}\right)^2 \alpha_k = 0 \Rightarrow \alpha_k = c_k e^{-D \left(\frac{k\pi}{L}\right)^2 t}$$

$$\Rightarrow U(x, t) = \sum_{k=1}^{\infty} c_k e^{-D \left(\frac{k\pi}{L}\right)^2 t} \cdot \cos\left(\frac{k\pi}{L}x\right) + c_0$$

$$\text{BV: } U(x, 0) = \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi}{L}x\right) + c_0 = M \delta\left(x - \frac{L}{2}\right)$$

Vi måste utveckla $M \delta\left(x - \frac{L}{2}\right)$ i en cos-serie!

$$M\delta(x - \frac{L}{2}) = \gamma_0 + \sum_1^{\infty} A_k \cos(\frac{k\pi}{L}x)$$

där $\gamma_0 = \frac{1}{L} \int_0^L M\delta(x - \frac{L}{2}) dx = \underline{\underline{\frac{M}{L}}}$ ↙ Vi har endast värde i $x = \frac{L}{2}$!

och $\beta_k = \frac{2}{L} \int_0^L M\delta(x - \frac{L}{2}) \cos(\frac{k\pi}{L}x) dx = \underline{\underline{\frac{2M}{L} \cos(\frac{k\pi}{2})}}$

Alltså: ~~$\frac{2M}{L} \cos(\frac{k\pi}{2})$~~

$$M\delta(x - \frac{L}{2}) = \frac{M}{L} + \frac{2M}{L} \sum_{k=1}^{\infty} \cos(\frac{k\pi}{2}) \cos(\frac{k\pi}{L}x)$$

Vi sätter likhet med $u(x, 0)$ och identifierar...

$$u(x, 0) = C_0 + \sum_1^{\infty} C_k \cos(\frac{k\pi}{L}x) = \frac{M}{L} + \frac{2M}{L} \sum_1^{\infty} \cos(\frac{k\pi}{2}) \cos(\frac{k\pi}{L}x)$$

$$\Rightarrow C_0 = \frac{M}{L}, \quad C_k = \cos(\frac{k\pi}{2}) \cdot \frac{2M}{L}$$

Vi har alltså lösningen:

$$u(x, t) = \frac{M}{L} + \frac{2M}{L} \sum_1^{\infty} \cos(\frac{k\pi}{2}) e^{-D(\frac{k\pi}{L})^2 t} \cdot \cos(\frac{k\pi}{L}x)$$

D.25

Ange $U[\varphi]$ om $U = \dots$ och $n = 3$

a) $U = \delta(x)\delta(y)\delta(z)$

$$U[\varphi] = (U | \varphi) = \iiint_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z) \varphi(x, y, z) dx dy dz =$$

$$= \varphi(0, 0, 0)$$

b) $U = \delta(x)\delta(y)$

$$U[\varphi] = (U | \varphi) = \iiint_{\mathbb{R}^3} \delta(x)\delta(y) \varphi(x, y, z) dx dy dz = \varphi(0, 0, z)$$

c) $U = \delta(x)$, pss: $U[\varphi] = \varphi(0, y, z)$

D.27

Lös 3D-potentialproblemet

$$\begin{cases} \Delta u = \delta(r-2) & , & 1 < r < 3 \\ u = 1 & & r = 1 \\ u = 3 & & r = 3 \end{cases}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Pga symmetri är $u = u(r)$, mha sfäriska får vi:

$$\frac{1}{r} (ru)'' = \delta(r-2) \Leftrightarrow (ru)'' = r\delta(r-2) = 2\delta(r-2)$$

$$(ru)' = \int 2\delta(r-2) dr = 2\theta(r-2) + A$$

$$ru = \int (2\theta(r-2) + A) dr = 2(r-2)\theta(r-2) + Ar + B$$

$$\Rightarrow U(r) = \frac{2}{r} (r-2) \Theta(r-2) + A + \frac{B}{r}$$

Randvillkor:

$$U(1) = 1 \Rightarrow \frac{2}{1} (1-2) \cdot 0 + \underline{A+B} = 1$$

$$U(3) = 3 \Rightarrow \frac{2}{3} (3-2) \Theta(3-2) + A + \frac{B}{3} = 3$$

$$\Leftrightarrow \frac{2}{3} + A + B/3 = 3 \Leftrightarrow \underline{3A+B=7}$$

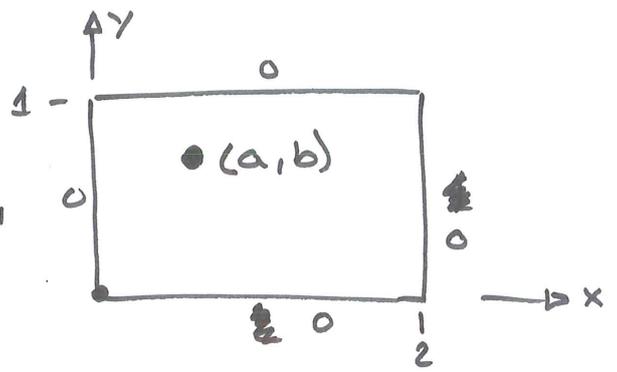
$$\begin{cases} A+B=1 \\ 3A+B=7 \end{cases} \Leftrightarrow \begin{cases} A+B=1 \\ 2A=6 \end{cases} \Leftrightarrow \begin{cases} A=3 \\ B=-2 \end{cases}$$

svar:

$$U(r) = \frac{2}{r} (r-2) \Theta(r-2) + 3 - \frac{2}{r}$$

D.29

Modellera



$$\begin{cases} -\Delta u = -u + \delta_{a,b}(x,y), & 0 < x < 2, 0 < y < 1 \\ u(0,y) = u(2,y) = 0 & 0 < y < 1 \\ u(x,0) = u(x,1) = 0 & 0 < x < 2 \end{cases}$$

~~Klassiska~~

$$Au = -\Delta u, \quad D_A = \{u \in C^2[(0,2) \times (0,1)] \mid u|_{\partial\Omega} = 0\}$$

Ansatz: $u(x,y) = X(x) \cdot Y(y)$

$$\text{PDE} \Rightarrow -\frac{X''}{X} - \frac{Y''}{Y} = \lambda \Rightarrow -\frac{X''}{X} = \mu, \quad -\frac{Y''}{Y} = \eta, \quad \lambda = \mu + \eta$$

Vi får två diff ekvationer:

$$\begin{cases} +X'' + \mu X = 0 \\ X(0) = X(2) = 0 \end{cases} \Rightarrow X(x) = \sin(\sqrt{\mu}x), \quad \mu_k = \left(\frac{k\pi}{2}\right)^2$$

$$\begin{cases} Y'' + \eta Y = 0 \\ Y(0) = Y(1) = 0 \end{cases} \Rightarrow Y(y) = \sin(\sqrt{\eta}y), \quad \eta = (j\pi)^2$$

$$\Rightarrow \varphi_{k,j}(x,y) = \sin\left(\frac{k\pi x}{2}\right) \cdot \sin(j\pi y), \quad k, j \in \mathbb{N}$$

$$\lambda_{k,j} = \left(\frac{k\pi}{2}\right)^2 + (j\pi)^2$$

Vi vill utveckla $\delta_{a,b}$ i $\{\varphi_{k,j}\}$

$$c_{k,j} = \frac{(\varphi_{k,j} | \delta_{a,b})}{(\varphi_{k,j} | \varphi_{k,j})} = \dots = 2 \sin\left(\frac{k\pi}{2} a\right) \sin(k\pi b)$$

$$\Rightarrow \delta_{a,b} = \sum_{\substack{k=1 \\ j=1}}^{\infty} c_{k,j} \varphi_{k,j}$$

Insättning i PDE (med $-\Delta u = Au = \lambda u$)

~~$\Delta u = Au = \sum_{k,j=1}^{\infty} \lambda_{k,j} u_{k,j} = \sum$~~ Ansats: $u(x,y) = \sum_1^{\infty} \alpha_{k,j} \varphi_{k,j}$

$$\Rightarrow \sum_1^{\infty} \lambda_{k,j} \cdot \alpha_{k,j} \cdot \varphi_{k,j} + \sum_1^{\infty} c_{k,j} \cdot \varphi_{k,j} = \sum_1^{\infty} 2 \sin\left(\frac{k\pi}{2} a\right) \sin(j\pi b) \varphi_{k,j}$$

$$\Rightarrow \alpha_{k,j} = \frac{2 \sin\left(\frac{k\pi}{2} a\right) \sin(j\pi b)}{\frac{k^2 \pi^2}{4} + j^2 \pi^2 + 1}$$

$$\Rightarrow u(x,y) = \sum_{k,j=1}^{\infty} \frac{2 \sin\left(\frac{k\pi}{2} a\right) \sin(j\pi b)}{\left(\frac{k\pi}{2}\right)^2 + (j\pi)^2 + 1} \sin\left(\frac{k\pi}{2} x\right) \sin(j\pi y)$$

D.30 Bestäm Fouriertransformen

$$a) \delta(t-1) \xrightarrow{\mathcal{F}} \mathcal{F}(\delta(t)) \cdot e^{-i\omega} = e^{-i\omega}$$

$$b) \delta'(t+2) \xrightarrow{\mathcal{F}} \hat{\delta}'(t) \cdot e^{2i\omega} = i\omega \cdot e^{2i\omega}$$

$$c) 2 \xrightarrow{\mathcal{F}} 2 \cdot 2\pi \delta(\omega) = 4\pi \delta(\omega)$$

$$d) \sin 2t = \frac{1}{2i} e^{2it} - \frac{1}{2i} e^{-2it} \xrightarrow{\mathcal{F}} \frac{1}{2i} 2\pi \delta(\omega-2) - \frac{1}{2i} 2\pi \delta(\omega+2) = \\ = \frac{\pi}{1} (\delta(\omega-2) - \delta(\omega+2))$$

$$e) (t+1)^2 = t^2 + 2t + 1 = t \cdot t + 2 \cdot t \cdot 1 + 1 \xrightarrow{\mathcal{F}}$$

$$\xrightarrow{\mathcal{F}} i^2 (2\pi \delta(\omega))'' + 2i (2\pi \delta(\omega))' + \delta(\omega) 2\pi =$$

$$= 2\pi (-\delta''(\omega) + 2i\delta'(\omega) + \delta(\omega))$$

f) $e^t \cdot \Theta(t)$ är inte tempererad

(Läs D.12 och D.13 på s. 382) och går
enl. def D.14 inte att Fouriertransformera.

D.31 Invers-Fouriertransformera.

a) $\delta(\omega - 1) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2\pi} e^{it}$

b) $\delta'(\omega + 2) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{i} t \cdot \frac{1}{2\pi} e^{-2it} = \frac{t}{2\pi i} e^{-2it}$

c) $2 \xrightarrow{\mathcal{F}^{-1}} 2\delta(t)$

d) $\sin 2\omega = \frac{e^{2i\omega} - e^{-2i\omega}}{2i} \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2i} (\delta(t-2) - \delta(t+2))$

e) $(\omega + 1)^2 = \omega^2 + 2\omega + 1 \xrightarrow{\mathcal{F}^{-1}} -\delta''(t) + \frac{2}{i}\delta'(t) + \delta(t) = -\delta''(t) - 2i\delta'(t) + \delta(t)$

D.34 Bestäm alla tempererade distributioner som satisfierar ekvationen.

$U * g(x) = \sin(x)$ där $g(y) = e^{-y^2}$, $y \in \mathbb{R}$

$e^{-y^2} \xrightarrow{\mathcal{F}} \sqrt{\pi} e^{-\omega^2/4}$, $\sin(x) \xrightarrow{\mathcal{F}} i\pi(\delta(\omega+1) - \delta(\omega-1))$

$U * g \xrightarrow{\mathcal{F}} \mathcal{F}(U) \cdot \sqrt{\pi} e^{-\omega^2/4} = i\pi(\delta(\omega+1) - \delta(\omega-1))$

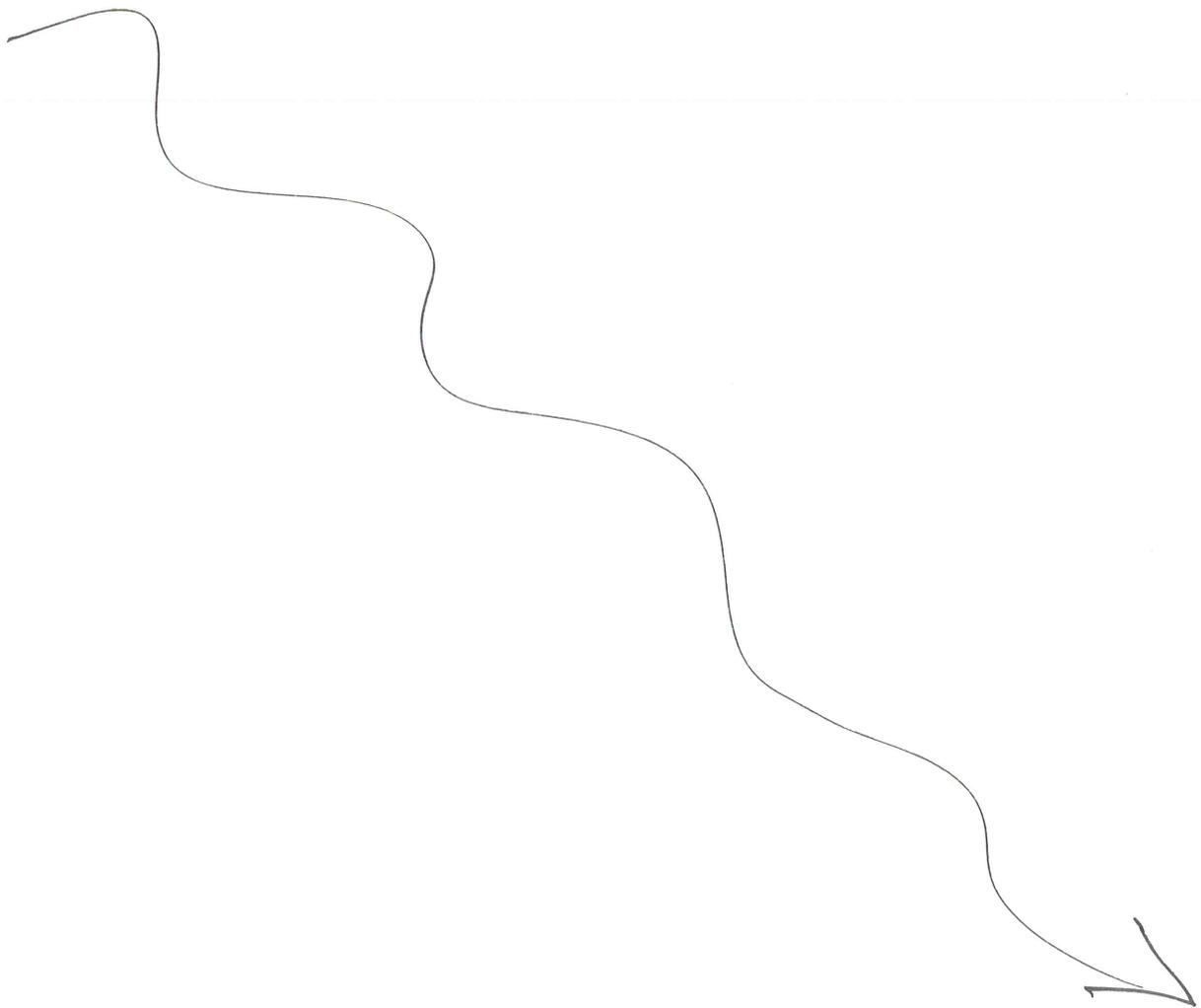
$\Rightarrow U = \mathcal{F}^{-1} \left(\frac{i\pi(\delta(\omega+1) - \delta(\omega-1))}{\sqrt{\pi} e^{-\omega^2/4}} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t + \omega^2/4} \cdot i\sqrt{\pi}(\delta_1 - \delta_{-1}) d\omega =$

$= \frac{i}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\omega t + \omega^2/4} (\delta_1 - \delta_{-1}) d\omega =$

$$= \frac{i}{2\sqrt{\pi}} (e^{-it + \frac{1}{4}} - e^{it + \frac{1}{4}}) = \frac{e^{1/4}}{\sqrt{\pi}} \cdot \frac{e^{it} - e^{-it}}{2i} =$$

$$= \frac{e^{1/4}}{\sqrt{\pi}} \cdot \sin(t).$$

Svar: $U(x) = \frac{e^{1/4}}{\sqrt{\pi}} \cdot \sin(x)$



D.36 Laplace

$$a) \delta(t-1) \xrightarrow{\mathcal{L}} e^{-s}$$

$$b) \delta'(t+2) \xrightarrow{\mathcal{L}} s e^{2s}$$

c) -

d) -

e) -

$$f) e^t \cdot \theta(t) \xrightarrow{\mathcal{L}} \frac{1}{s-1} \quad \sigma > 1$$

D.41 Utvidga diff.ekvationerna nedan till hela \mathbb{R} så att randvillkoren uppfylls.

$$a) \begin{cases} u''(x) - iu(x) = 0, & x > 0 \\ u(0) = 1 \end{cases}$$

$$\Leftrightarrow (u^-)'' - i(u^-) = 2\delta(x), \quad x \in \mathbb{R}$$

$$b) \begin{cases} u'' - iu = 0, & x > 0 \\ u'(0) = 1 \end{cases}$$

$$\Leftrightarrow (u^+)'' - iu^+ = \delta(x), \quad x \in \mathbb{R}$$