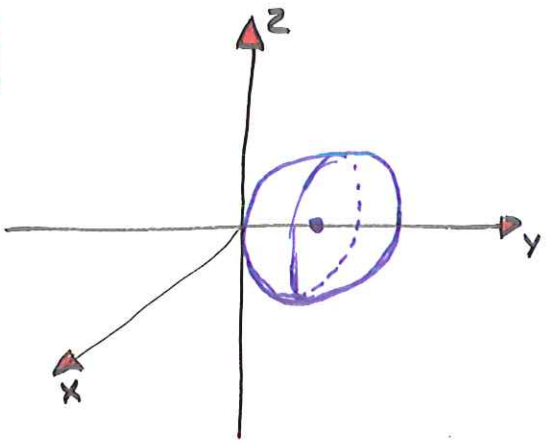
7. Integralsatser

$\boxed{7.1}$ Beräkna $\iint_S (2x + x^3 z)\bar{n}\, dS$:

$$= \Big/ \operatorname{grad}(2x + x^3 z) = (2 + 3x^2 z,\ 0,\ x^3) \Big/ =$$



$$= \iiint_V (2 + 3x^2 z,\ 0,\ x^3)\, dV =$$

udda

Vi ser här att vi har symmetri x- och z-led.

$$= \iiint_V (2,0,0)\, dV = \boxed{\frac{4\pi}{3}(2,0,0)}$$

$\boxed{7.2}$ Visa att

a) $\int_S f(\nabla \times \bar{A})\, d\bar{a} = \int_S [\bar{A} \times (\nabla f)] \cdot d\bar{a} + \oint_P f \bar{A}\, d\bar{l}$

Enligt FS: $\nabla \times (f\bar{A}) = f(\nabla \times \bar{A}) - A \times (\nabla f)$ (1)

Gauss universalsats: $\int_S \nabla \times (fA)\, d\bar{S} = \oint_P f\bar{A}\, dl$ (2)

Vi integrerar (1) och identifierar med (2)

$$\Rightarrow \oint_P f\bar{A}\, d\bar{l} = -\int_S [A \times (\nabla f)]\, d\bar{a} + \int_S f(\nabla \times A)\, d\bar{a}$$

#

b) Visa att

$$\boxed{\int_V \bar{B}\cdot(\nabla\times\bar{A})\,d\tau = \int_V \bar{A}\cdot(\nabla\times\bar{B})\,d\tau + \oint_S \bar{A}\times\bar{B}\,d\bar{a}}$$

Enligt FS: $\nabla\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\times\bar{B})$

Gauss universalsats:

$$\iiint_V B\cdot(\nabla\times A)\,dV = \iiint_V \bar{A}\cdot(\nabla\times\bar{B})\,dV + \iiint_V \nabla\cdot(\bar{A}\times\bar{B})\,dV =$$

$$= \iiint_V \bar{A}\cdot(\nabla\times\bar{B})\,dV + \oint\!\!\oint_S (A\times B)\,d\bar{S}$$

#

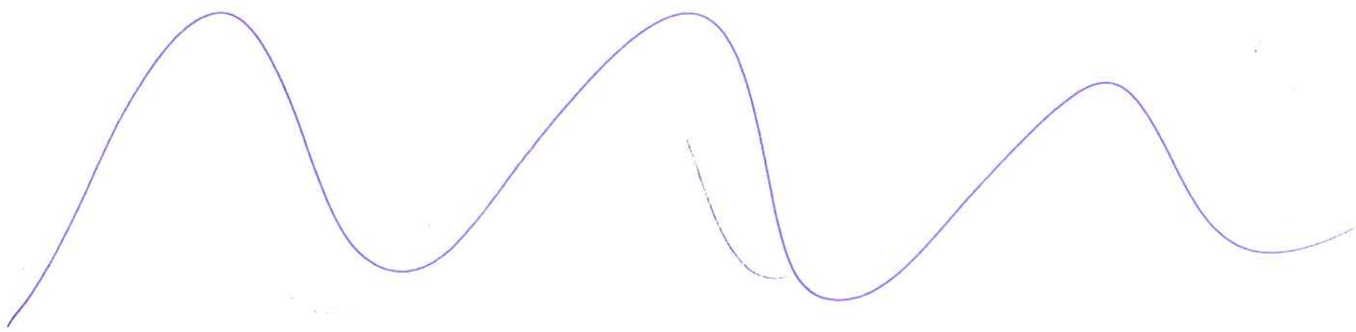
7.3 Beräkna integralen

$$\frac{1}{2}\oiint_S d\bar{S}\times(\bar{a}\times\bar{r}) = \frac{1}{2}\iiint_V \nabla\times(\bar{a}\times\bar{r})\,dV = \frac{1}{2}\iiint_V \Big(\cancel{(\bar{r}\cdot\nabla)\bar{a}} - \underbrace{(\bar{a}\cdot\nabla)\bar{r}}_{=a} + \underbrace{\bar{a}(\nabla\cdot\bar{r})}_{=3\bar{a}} - \cancel{\bar{r}(\nabla\cdot a)}\Big)dV = \frac{1}{2}\iiint_V 2\bar{a}\,dV = \boxed{\bar{a}\cdot V}$$

7.4 Beräkna ytintegralen.

$$\oiint_S (a\times r)\times d\bar{S} = -\iiint_V \nabla\times(a\times r)\,dV =$$

$$= -\iiint_V 2\bar{a}\,dV = -2\bar{a}V = 2\cdot\bar{a}\cdot\frac{4\pi}{3} = \boxed{\frac{8\pi}{3}\bar{a}}$$



7.5 Bestäm alla vektorfält $\bar{A}$ för vilka
$$\oiint_S \bar{A}\cdot\bar{n}\,dS = 7\cdot V$$

$$\oiint_S \bar{A}\cdot\bar{n}\,dS = \iiint_V \nabla\cdot A\,dV = 7V$$

Eftersom $\iiint_V dV = V$ så måste $\nabla\cdot A = 7$ vilket är en diffekvation med lösning:

$$A = A^h + A^p$$

Homogen lösning

$$\nabla\cdot A = 0$$

$$\Leftrightarrow \nabla\cdot(\nabla\times\bar{v}) = 0$$ (Detta stämmer för alla ~~skalär~~ vektorfält $\bar{v}$)

$$\Rightarrow \boxed{A^h = \nabla\times\bar{v}}$$

Partikulärlösning

$$\nabla\cdot A^p = 7$$

$$\Leftrightarrow \boxed{A^p = (2x, 3y, 2z)}$$ (vi kan välja A^p till massa olika.

$$\Rightarrow \boxed{A = \nabla\times\bar{v} + (2x, 3y, 2z)}$$