

5. GAUSS SATS

5.1

Beräkna $\operatorname{div}(\vec{F})$ för

a) $\vec{F} = (x, y, z) = r$

$$\operatorname{div}(\vec{F}) = 1 + 1 + 1 = 3$$

b) $\vec{F} = (\sin(xy), e^{2xy}, \cos^2(2x))$

$$\operatorname{div}(\vec{F}) = y \cos(xy) + 2x e^{2xy} - 2x \cos(2x) \sin(x).$$

c) $\vec{F} = \operatorname{grad}(\phi), \phi = 3xyz^3$

$$\operatorname{grad}(\phi) = (3yz^3, 3xz^3, 9xy^2z^2)$$

$$\operatorname{div}(\operatorname{grad}(\phi)) = 0 + 6xz^3 + 18xy^2z$$

5.2

Beräkna $\operatorname{div}(\vec{F})$

$$\vec{F} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{1/2}}$$

Efter lite derivering får vi:

$$\frac{dF_x}{dx} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2 + z^2}}}$$

Symmetri

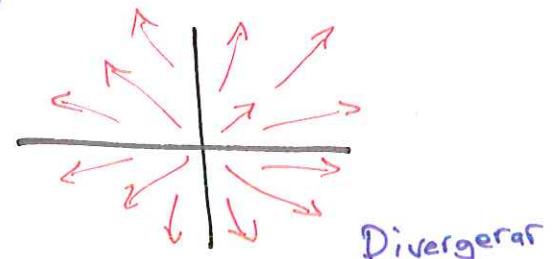
$$\operatorname{div}(\vec{F}) = \frac{3}{\sqrt{1 - \frac{x^2 + y^2 + z^2}{y^2 + z^2}}} - \frac{x^2 + y^2 + z^2}{(y^2 + z^2)^{3/2}}$$

5.3

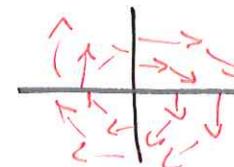
Beräkna $\operatorname{div}(A)$ och $\operatorname{div}(B)$

$$A = (x, y, 0), B = (y, -x, 0)$$

$$\operatorname{div}(A) = 1 + 1 + 0 = 2 \neq 0$$



$$\operatorname{div}(B) = 0 + 0 + 0 = 0$$



Divergerar inte

5.4 Kontrollera Gauss sats

$$\operatorname{div}(\vec{v}) = y + 2z + 3x$$

$$\iint_V \operatorname{div}(\vec{v}) dV = \iiint_0^2 0^2 0^2 (y + 2z + 3x) dx dy dz = 48$$

Botten ($z=0$)

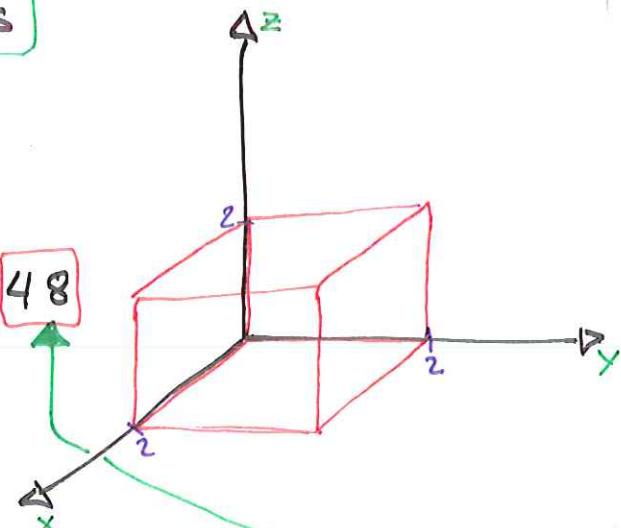
$$\iint_S (-3zx) dx dy = 0$$

Vänster sida ($y=0$)

$$\iint_S (-2yz) dx dz = 0$$

Baksidan

$$\iint_S (-xy) dy dz = 0$$

Toppen ($z=2$)

$$\iint_0^2 3zx dx dy = 24$$

Höger sida ($y=2$)

$$\iint_0^2 2yz dx dz = 16$$

Framsidan $x=2$

$$\iint_0^2 xy dy dz = 8$$

48

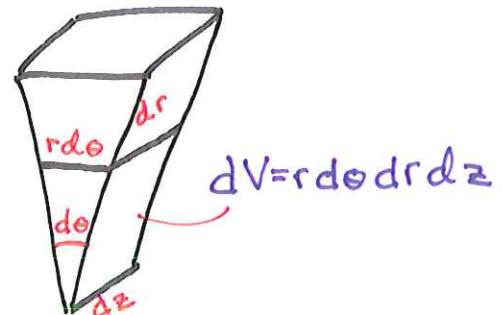
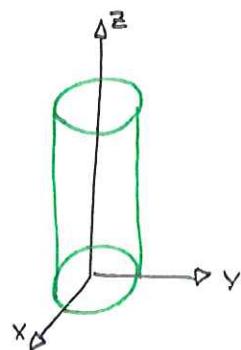
5.5 Beräkna flödet genom ytan

$$\bar{F} = (y^2x, x^2y, z^2)$$

$$\operatorname{div}(\bar{F}) = y^2 + x^2 + 2z$$

Gauss sats

$$\iint_S \bar{F} d\bar{s} = \iiint \operatorname{div}(\bar{F}) dV = \iiint (y^2 + x^2 + 2z) dV =$$



$$= \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} = \int_0^2 \int_0^{2\pi} \int_0^2 (r^2 \sin^2 \theta + r^2 \cos^2 \theta + 2z) r d\theta dr dz =$$

$$= \int_0^2 \int_0^{2\pi} \int_0^2 (r^2 + 2z) r d\theta dr dz = 2\pi \int_0^2 \int_0^2 (r^3 + 2rz) dr dz =$$

$$= \int_0^2 (2r^3 + 4r) dr = \boxed{32\pi}$$

5.6

Beräkna största värdet $\iint_S \bar{B} d\bar{s}$ antar.

$$\bar{B} = (x - x^3, y - y^3, z - z^3) \cdot \frac{1}{3}$$

$$\operatorname{div}(\bar{B}) = 1 - (x^2 + y^2 + z^2) = 1 - r^2$$

endast avståndet till origo gör skillnad.

Gauss sats

$$\iint_S \bar{B} d\bar{s} = \iiint_V \operatorname{div}(\bar{B}) dV = \iiint_V (1 - r^2) r^2 \sin\theta dr d\theta d\varphi =$$

$$= 4\pi \left(\frac{R^3}{3} - \frac{R^5}{5} \right)$$

$$\text{Derivering: } 4\pi(R^2 - R^4) = 0 \Rightarrow$$

$$R = 1$$

kontrollera att detta är ett maximum genom att derivera en gång till

5.7

Beräkna flödet

$$\mathbf{A} = (x^2, 2y, z), \quad \bar{r} = R(\sin u \cos v, \sin u \sin v, \cos u)$$

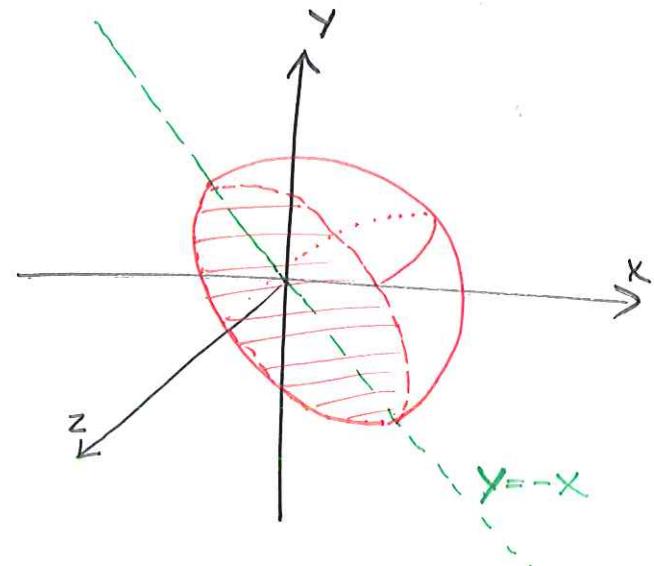
$$\iint_S \mathbf{A} d\bar{a} = \iiint_V \operatorname{div}(\mathbf{A})$$

5.8

Beräkna $\iint_S \bar{F} d\bar{s}$

$$\bar{F} = (x^3, y^3, z^3)$$

$$S: \begin{cases} x^2 + y^2 + z^2 = R^2 \\ x + y = 0 \end{cases}$$



$$\iint_S (x^3, y^3, z^3) d\bar{s} = \iiint_V \operatorname{div}(x^3, y^3, z^3) dV =$$

$$= \iiint_V 3(x^2 + y^2 + z^2) dV = \left/ \begin{array}{l} \text{Eftersom } \bar{F} \text{ är symmetrisk} \\ \text{kan vi ta halva integralen} \\ \text{över hela sfären } V. \end{array} \right/ =$$

$$= \frac{3}{2} \iiint_V (x^2 + y^2 + z^2) dV = \left/ \begin{array}{l} \text{övergång till sferiska koordinater} \\ x^2 + y^2 + z^2 = r^2 \end{array} \right/ =$$

$$= \frac{3}{2} \iiint_V (r^2) r^2 \sin\theta dr d\theta d\phi = \frac{3}{2} \int_0^{2\pi} \int_0^{\pi} \int_0^R (r^4) \sin\theta dr d\theta d\phi =$$

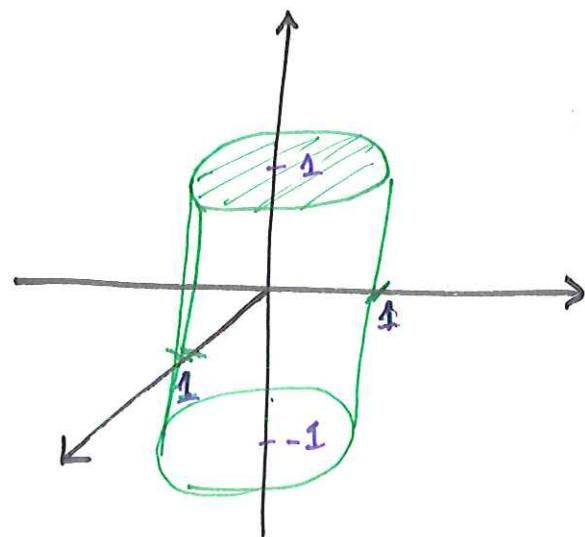
$$= \boxed{\frac{6\pi R^5}{5}}$$

5.9

Beräkna flödet

$$A = (xz^2, 2xy, z^2+2)$$

$$\begin{cases} x^2+y^2=1 \\ z: -1 \rightarrow 1 \end{cases}$$



$$\iiint_S A \cdot d\bar{s} = \iiint_V \operatorname{div}(A) dV = \iiint_V (z^2 + 2x + 2z) dV =$$

$$= \iiint_V (z^2 + 2z + 2r\cos\theta) r dr d\theta dz =$$

$$= \iiint_V z^2 r + 2zr + 2r^2 \cos\theta d\theta dr dz =$$

$$= \iint \left[(z^2 r + 2zr) \theta + 2r^2 \cancel{\sin\theta} \right]_{0}^{2\pi} dr dz =$$

$$= \iint 2\pi (z^2 r + 2zr) dr dz = \int_{-1}^1 2\pi \left[\frac{1}{2} z^2 r^2 + zr^2 \right]_0^1 dz =$$

$$= \int_{-1}^1 2\pi \left(\frac{1}{2} z^2 + z \right) dz = \left[\frac{2\pi}{2} \left(\frac{1}{3} z^3 + \frac{1}{2} z^2 \right) \right]_{-1}^1 = 2\pi \left(\frac{1}{3} - \frac{1}{2} - \left(-\frac{1}{3} + \frac{1}{2} \right) \right) =$$

$$= \boxed{\frac{2\pi}{3}}$$

Detta går att kontrollera genom att dela upp ytan i tre delar: sida, topp, botten.