

KAPITEL 4

4.1

Lös varmeledningsproblem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & , t > 0, x \in \mathbb{R} \\ u(x, 0) = e^{-x^2} & , x \in \mathbb{R} \end{cases}$$

Vi använder Fouriertransform i x-led.

$$\begin{cases} U_t - i s^2 U = 0 \\ U(s, 0) = \sqrt{\pi} e^{-s^2/4} \end{cases} \Leftrightarrow \begin{cases} U = \hat{g}(s) \cdot e^{-s^2 t} \\ U(s, 0) = \sqrt{\pi} e^{-s^2/4} \end{cases}$$

$$\Rightarrow U(s, 0) = \sqrt{\pi} e^{-s^2/4} = \hat{g}(s) \cdot e^0 = \hat{g}(s)$$

$$\text{Vi har alltså: } U = \sqrt{\pi} \cdot e^{-s^2 t - \frac{s^2}{4}} = \sqrt{\pi} e^{-\frac{s^2}{4}(1-4t)}$$

~~Detta är fel~~

Invers Fourier:

(Regel 13 & skalning)

$$U = \frac{1}{\sqrt{4t+1}} e^{-x^2/(4t+1)}$$

4.3

Lös värmelämningsproblemet

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = -\alpha u, \quad t > 0, \quad x \in \mathbb{R} \\ u(x, 0) = \delta(x), \quad x \in \mathbb{R} \\ u(x, t) \rightarrow 0 \text{ då } x \rightarrow \pm \infty, \quad t > 0 \end{array} \right.$$

Tolkning: Vi har en punktvärme "i mitten" av en oändlig 1D-stång vid $t=0$. Värmen sprids. Kalla ändar.

F i x-led:

$$\left\{ \begin{array}{l} U_t' + \alpha s^2 U = -\alpha U \\ U(s, 0) = 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} U_t' + (\alpha + \alpha s^2) U = 0 \\ U(s, 0) = 1 \end{array} \right.$$

alla $s \neq 0$ riktas

$$\Rightarrow \left\{ \begin{array}{l} U = \hat{g}(s) \cdot e^{-(\alpha + \alpha s^2)t} \\ U(s, 0) = \hat{g}(s) = 1 \end{array} \right. \quad \left\{ \begin{array}{l} u = F^{-1}(e^{-\alpha t} \cdot e^{-\alpha s^2 t}) = \\ = e^{-\alpha t} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{4\alpha t}} e^{-x^2/4\alpha t} \end{array} \right.$$

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$$\begin{cases} U_t' - iU_{xx}'' = 0, \quad t > 0, \quad x \in \mathbb{R} \\ U(x, 0) = \sin^2 x, \quad x \in \mathbb{R} \end{cases}$$

Fourier i x-led: $\Rightarrow = \left(\frac{1}{2i}\right)^2 (e^{ix} - e^{-ix})^2 = -\frac{1}{4} (e^{2ix} - 2 + e^{-2ix})$

$$\begin{cases} U_t' - i(is)^2 U = 0 \\ U(s, 0) = \sqrt{\frac{\pi}{2}} \left(-\frac{1}{2} \delta(s-2) + \delta(s) - \frac{1}{2} \delta(s+2) \right), \end{cases}$$

$$\Leftrightarrow U = \hat{g}(s) e^{-is^2 t}, \quad g(s) = -\frac{\pi}{2} (\delta_2 - 2\delta + \delta_2)$$

Allså

$$U = -\frac{\pi}{2} (\delta_2 - 2\delta + \delta_2) e^{-i\omega^2 t}$$

Invers Fourier ger:

$$u(x, t) = \frac{1}{2} (1 - e^{-4it} \cos 2x)$$

4.6

Lös problemet

PDE $\begin{cases} U_{tt} - U_{xx} = 0 & , t > 0, x \in \mathbb{R} \\ U(x, 0) = 0, U_t(x, 0) = \delta(x) & , x \in \mathbb{R} \end{cases}$

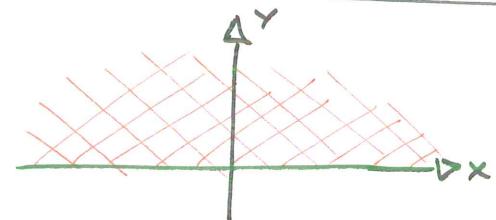
Lösningen ges av d'Alemberts formel där

$c = 1, g(x) = 0, h(x) = \delta(x)$ (se s.139)

$$\Rightarrow U(x, t) = \frac{1}{2} \int_{x-t}^{x+t} \delta(y) dy = \boxed{\frac{1}{2} (\Theta(x+t) - \Theta(x-t))}$$

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$$\begin{cases} U_{xx}'' + U_{yy}'' = 0 & , y > 0, x \in \mathbb{R} \\ U(x, 0) = \frac{1}{1+x^2} \end{cases}$$



Fourier i x-led:

$$\begin{cases} (i\omega)^2 U + U_{yy}'' = 0 \\ U(x, 0) = \pi e^{-|w|} \end{cases} \Leftrightarrow \begin{cases} g_1(\omega) e^{wy} + g_2(\omega) e^{-wy} = 0 \\ U(x, 0) = \pi e^{-|w|} \end{cases}$$

$$\Rightarrow g_1 + g_2 = \pi e^{-|w|}$$

Vi vill att lösningen ska vara begränsad:

$$g_1(\omega) e^{i\omega y} \rightarrow \infty \text{ då } y \rightarrow \infty \text{ om } \omega > 0$$

$$g_2(\omega) e^{-i\omega y} \rightarrow \infty \text{ då } y \rightarrow \infty \text{ om } \omega < 0$$

$$\Rightarrow g_1 = \begin{cases} 0 & \omega > 0 \\ \pi e^{-|\omega|} & \omega < 0 \end{cases}, g_2(\omega) = \begin{cases} \pi e^{|\omega|} & \omega > 0 \\ 0 & \omega < 0 \end{cases}$$

Alltså har vi

$$U(\omega, y) = \pi e^{-|\omega|} \cdot e^{-|\omega| \cdot y} = \pi e^{-|\omega|(y+1)}$$

Invers Fourier.

~~Uppgift~~ vi betecknar $a = \frac{1}{y+1} < 0$

$$\pi e^{-|\omega|} \xrightarrow{\mathcal{F}^{-1}} \frac{1}{1+x^2}$$

$$a \cdot \frac{1}{|\omega|} \cdot \pi e^{-|\omega| \frac{a}{|\omega|}} \xrightarrow{\mathcal{F}^{-1}} \frac{a}{1+(ax)^2} = \frac{1}{y+1} \cdot \frac{1}{1+\frac{x^2}{(y+1)^2}} =$$

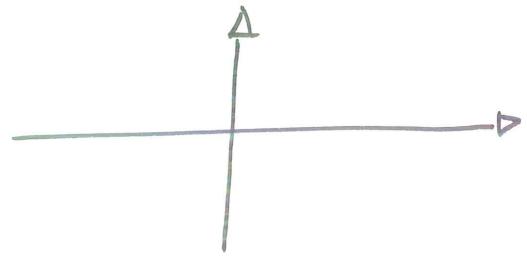
$$= \frac{(y+1)^2}{(y+1)^3((y+1)^2+x^2)} = \cancel{\frac{1}{(y+1)^3((y+1)^2+x^2)}} = \boxed{\frac{1}{(y+1)((y+1)^2+x^2)}}$$

Nära nog.

4.10

Lös värmeledningsproblem

$$\begin{cases} U_t' - U_{xx}'' = 0 & , t > 0, x > 0 \\ U(x, 0) = \delta(x-1), & x > 0 \end{cases}$$



a) $U(0, t) = 0, t > 0$

Vi gör en ensidig Laplace i t-led, NES.

Udda utveckning:

$$\bar{U}(x, t) = \begin{cases} U(x, t), & x > 0 \\ -U(-x, t), & x < 0 \end{cases}$$

Enligt speglingslemmet får vi:

$$\frac{\partial^2 \bar{U}}{\partial x^2} = \left(\frac{\partial^2 U}{\partial x^2} \right)^- + 2u(0, t) \delta'(x)$$

$$\frac{\partial \bar{U}}{\partial t} = \left(\frac{\partial U}{\partial t} \right)^-$$

$$\Rightarrow \bar{U}_t' - \bar{U}_{xx}'' = (U_t' - U_{xx}'')^- - 2u(0, t) \delta'(x) = 2 \cdot 0 \cdot \delta'(x) = 0$$

Laplace i x-led med $\mathcal{L}_x \bar{U} = U$ ger:

$$\frac{\partial U}{\partial t} - s^2 U = 0 \Rightarrow U = g(s) \cdot e^{s^2 t}$$

~~Laplace~~ av $\mathcal{B}V$ ger:

$$\begin{cases} U(s, 0) = e^{-s} \\ U(s, t) = g(s) \cdot e^{s^2 t} \Rightarrow g(s) = e^{-s} \end{cases}$$

$$\Rightarrow U(s, t) = e^{-s} \cdot e^{s^2 t}$$

Om vi kaller $e^{s^2 t} = F(s)$ så vet vi att

$$e^{-s} \cdot F(s) \xrightarrow{\mathcal{Z}^{-1}} f(x - 1)$$

Vi vet också att:

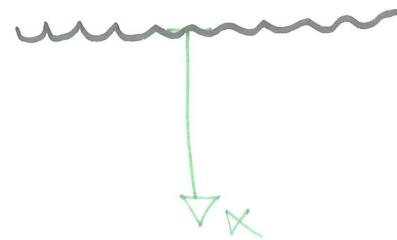
$$\sqrt{s!} e^{s^2/4} \xrightarrow{\mathcal{Z}^{-1}} e^{-x^2}$$

$$\Rightarrow \frac{4}{\pi} \cdot \frac{\sqrt{\pi}}{\sqrt[4]{\pi}} \cdot e^{s^2 t} \xrightarrow{\mathcal{Z}^{-1}} \frac{4}{\pi} \cdot e^{-(\frac{4}{\pi} t)^2}$$

$$\Rightarrow \cancel{e^{s^2 t}} e^{-s} \xrightarrow{\mathcal{Z}^{-1}} \frac{4}{\sqrt{\pi t}} e^{-(\frac{4}{\pi}(x-1))^2} \quad ?$$

4.15 Ställ upp en mattemodell

$$\begin{cases} U_t' - D U_{xx}'' = 0, \quad x > 0, \quad t > 0 \\ U_x'(0,t) = 0 \\ U(x,0) = q_0 \cdot \delta(x) \end{cases}$$



b) Lös

Vi utvidgar jämt till hela \mathbb{R} .

$$\delta^+(x) = 2\delta(x)$$

$$U^+(x,t) = \begin{cases} U(x,t), & x > 0 \\ U(-x,t), & x < 0 \end{cases}$$

$$\frac{\partial U^+}{\partial t} = \left(\frac{\partial U}{\partial t} \right)^+, \quad \frac{\partial^2 U^+}{\partial x^2} = \left(\frac{\partial^2 U}{\partial x^2} \right)^+ + 2U'(0,t)\delta(x)$$

$$\Rightarrow U_t' - D U_{xx}'' = (U_t' - D U_{xx}'')^+ + 2U'(0,t)\delta(x) = 2 \cdot 0 \cdot \delta = 0$$

Laplace i x-led. med $\mathcal{L}_x U^+ = U$ ger:

$$\begin{cases} U_t' - D s^2 U = 0 \\ U(s,0) = \mathcal{L}(2\delta(x)) = 2 \end{cases}$$

Det blir säkert rätt.

$$\Rightarrow U(s,t) = 2 e^{Ds^2 t} \Rightarrow U(x,t) = 2q_0 \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}, \quad x > 0$$

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Ett nytt sjöproblem.

PDE $\left\{ \begin{array}{l} U_t' - D U_{xx}'' = 0 \\ , x > 0, t > 0 \end{array} \right.$

RV $\left\{ \begin{array}{l} U(0,t) = k \\ \end{array} \right.$

BV $\left\{ \begin{array}{l} U(x,0) = 0 \end{array} \right.$

$$\begin{aligned} V = U &\Rightarrow V_t' - D V_{xx}'' = U_t' - D(U_{xx}'' + 2U(0,t)\delta') = \\ &= (U_t' - D U_{xx}'') - D2k\delta' = \\ &= -D2k\delta' \end{aligned}$$

Ny PDE: $V_t' - D V_{xx}'' = -D2k\delta'$

Ensidig Laplace ger i x-led ger:

$$V_t' - s^2 D V = -2Dk s \Rightarrow V(s,t) = \underbrace{A(s)e^{Ds^2 t}}_{\text{homogen}} + \underbrace{\frac{2k}{s}}_{\text{partikulär}}$$

$$\text{BV} \Rightarrow V(s,0) = A(s) + \frac{2k}{s} = 0$$

$$\Rightarrow A(s) = -\frac{2k}{s}$$

$$\Rightarrow V(s,t) = \frac{2k}{s} (1 - e^{Ds^2 t})$$

$$V(x,t) = \mathcal{L}^{-1}(V(s,t)) = \mathcal{L}^{-1}\left(\frac{2k}{s}\right) * \mathcal{L}^{-1}(1 - e^{Ds^2 t})$$

$$\mathcal{L}^{-1}\left(\frac{2k}{s}\right) = 2k\Theta(x)$$

$$\mathcal{L}^{-1}(1 - e^{Ds^2t}) = \delta - \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$$

$$\Rightarrow V(x,t) = 2k \int_{-\infty}^{\infty} \Theta(x-y) \cdot \left(\delta(y) - \frac{1}{\sqrt{4\pi Dt}} e^{-y^2/4Dt} \right) dy =$$

$$= 2k\Theta(x) - \int_{-\infty}^x \frac{2k}{\sqrt{4\pi Dt}} e^{-y^2/4Dt} dy = \left| \begin{array}{l} m = \frac{y}{\sqrt{4Dt}} \\ dm = \frac{1}{\sqrt{4Dt}} dy \end{array} \right|$$

$$= 2k\Theta(x) - \frac{2k}{\sqrt{4\pi Dt}} \int_{-\infty}^{x/\sqrt{4Dt}} e^{-m^2} dm =$$

$$= 2k\Theta(x) - \frac{2k}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(m) \right]_{-\infty}^{x/\sqrt{4Dt}} =$$

$$= 2k\Theta(x) - k \left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{4Dt}}\right) \right)$$

För $x > 0 \Rightarrow \Theta(x) = 1$

svar V(x,t) = k \left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{4Dt}}\right) \right)

4.17

Lös problemet

PDE $\left\{ \begin{array}{l} u'_t - u''_{xx} = 0, \quad x > 0, t > 0 \\ u'_x(0,t) = 1, \quad t > 0 \\ u(x,0) = 0, \quad x > 0 \end{array} \right.$

RV $\left\{ \begin{array}{l} u'_x(0,t) = 1, \quad t > 0 \\ u(x,0) = 0, \quad x > 0 \end{array} \right.$

Vi börjar med att homogenisera RV.

$$v(x,t) = u(x,t) - x$$

Ustat blir = 1 vid derivering!

$\left\{ \begin{array}{l} v'_t - v''_{xx} = 0, \quad x > 0, t > 0 \\ v'_x(0,t) = 0, \quad t > 0 \\ v(x,0) = -x, \quad x > 0 \end{array} \right.$

Vi har neuman-villkor och måste df speglas jämt!

$$v^+ = \begin{cases} v & \text{då } x > 0 \\ v(-x,t) & \text{då } x < 0 \end{cases}$$

Speglingsslemmat ger;

PDE $\left\{ \begin{array}{l} v^{'t} - v^{''}_{xx} = 0 \\ v^+(x,0) = -x\Theta(x) + x(1-\Theta(x)) = \underbrace{x(1-2\Theta(x))}_g, \quad x \in \mathbb{R}, t > 0 \end{array} \right.$

BV $v^+(x,0) = -x\Theta(x) + x(1-\Theta(x)) = \underbrace{x(1-2\Theta(x))}_g, \quad x \in \mathbb{R}$

Till denna har vi lösningen:

$$v^+(x,t) = \int_{-\infty}^{\infty} g(x-\alpha) G(\alpha,t) d\alpha + \cancel{\int_{\partial\Omega}}$$

Vi har alltså:

$$v^+(x,t) = -\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} (x-\alpha)(1-2\Theta(x-\alpha)) e^{-\alpha^2/4t} d\alpha =$$

$$= \frac{1}{\sqrt{4\pi t}} \left(- \int_{-\infty}^x (x-\alpha) e^{-\alpha^2/4t} d\alpha + \int_x^{\infty} (x-\alpha) e^{-\alpha^2/4t} d\alpha \right) =$$

$$= \frac{1}{\sqrt{4\pi t}} \cdot \left(-x \int_{-\infty}^x e^{-\alpha^2/4t} d\alpha - \int_{-\infty}^x \alpha \cdot e^{-\alpha^2/4t} d\alpha + x \int_{-\infty}^x e^{-\alpha^2/4t} d\alpha + \int_x^{\infty} \alpha \cdot e^{-\alpha^2/4t} d\alpha \right) =$$

$$= \frac{1}{\sqrt{4\pi t}} \left(x \int_{-\infty}^{\infty} e^{-\alpha^2/4t} d\alpha - \int_{-\infty}^x e^{-\alpha^2/4t} d\alpha \right) + \dots \text{ FUCK NO } \dots =$$

$$= -\frac{2\sqrt{t}}{\sqrt{\pi}} e^{-x^2/4t} - x \cdot \operatorname{erf}\left(\frac{x}{\sqrt{4t}}\right).$$

$$\Rightarrow u(x,t) = x + v^+ = x - \frac{2\sqrt{t}}{\sqrt{\pi}} e^{-x^2/4t} - x \cdot \operatorname{erf}\left(\frac{x}{\sqrt{4t}}\right)$$

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Värmeleddning med periodiskt varierande temp

PDE $\left\{ \begin{array}{l} U_t - a U_{xx} = 0, \quad x > 0, t > 0 \\ RV \quad u(0,t) = A \cdot \sin(\omega t), \quad t > 0 \end{array} \right.$

a) Bestäm den stationära periodiska lösningen.

Ansats: $u(x,t) = H(x) \cdot e^{i\omega t}$ (se s. 125)

PDE: $H(x) \cdot i\omega \cdot e^{i\omega t} - a H''(x) e^{i\omega t} = 0$

$$\Rightarrow H \cdot i\omega - a \cdot H'' = 0$$

$$\Rightarrow H'' - \frac{i\omega}{a} H = 0 \quad \cancel{\text{Detta är en matematiskt svår lösning}}$$

$$\Rightarrow H(x) = C_1 \cdot e^{\sqrt{\frac{i\omega}{a}} x} + C_2 e^{-\sqrt{\frac{i\omega}{a}} x}$$

RV $\Rightarrow H(0) = A$, $H(x)$ begr. $\Rightarrow \underline{C_1 = 0}$.

$$\Rightarrow H(0) = A = C_2 \cdot 1 \Rightarrow \underline{C_2 = A}$$

$$\Rightarrow u(x,t) = A \cdot e^{-\sqrt{\frac{i\omega}{a}} x} \cdot e^{i\omega t}$$

Mha principalgrenen (funktions teori) får vi:

$$u(x,t) = A \cdot e^{-i(\omega t - \sqrt{\frac{\omega}{2a}} x)} \cdot e^{-\sqrt{\frac{\omega}{2a}} x}$$

Vi söker endast imaginärdelen

$$\Rightarrow u(x,t) = A e^{-\sqrt{\frac{\omega}{2a}}} \cdot \sin(\omega t - \sqrt{\frac{\omega}{2a}} x)$$

b) Bestäm inträngningsdjupet.

$$\frac{A}{e} = A e^{-\sqrt{\frac{\omega}{2a}} \cdot x} \Rightarrow x = \sqrt{\frac{2a}{\omega}}$$

c) Sätt in siffror, jag heter siffror.