

KAPITEL 3

3.2 Bestäm Fourierkoefficienterna

a) $x = \sum_{k=0}^{\infty} a_k \cdot \cos(k\pi x) , 0 < x < 1$

FS: $x = C_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi x)$

$$C_0 = \frac{1}{1} \int_0^1 x dx = \boxed{\frac{1}{2}}$$

$$\begin{aligned} a_k &= \frac{2}{1} \int_0^1 x \cdot \cos(k\pi x) dx = 2 \left[x \cdot \frac{\sin(k\pi x)}{k\pi} \right]_0^1 - 2 \int_0^1 \frac{\sin(k\pi x)}{k\pi} dx = \\ &= 0 + 2 \left[\frac{\cos(k\pi x)}{k^2\pi^2} \right]_0^1 = \frac{\cos(k\pi) - \cos(0)}{k^2\pi^2} = \boxed{2 \frac{(-1)^k - 1}{k^2\pi^2}} \end{aligned}$$

b) $x = \sum_{k=1}^{\infty} B_k \sin(k\pi x) , 0 < x < 1$

$$\begin{aligned} B_k &= \frac{2}{1} \int_0^1 x \cdot \sin(k\pi x) dx = 2 \left[-x \cdot \frac{\cos(k\pi x)}{k\pi} \right]_0^1 + 2 \int_0^1 \frac{\cos(k\pi x)}{k^2\pi^2} dx = \\ &= 2 \left(\frac{(-1)^{k+1}}{k\pi} \right) + 2 \left[\frac{\sin(k\pi x)}{k^2\pi^2} \right]_0^1 = \boxed{\frac{2(-1)^{k+1}}{k\pi}} \end{aligned}$$

3.5

Lös värmelämningsproblem

$$\left\{ \begin{array}{l} U_t - U_{xx} = 0, \quad 0 < x < 1, \quad t > 0 \\ U(0, t) = U(1, t) = 0 \quad t > 0 \\ U(x, 0) = \sin \pi x + 2 \sin 3\pi x \quad 0 < x < 1 \end{array} \right.$$

PDE
RV
BV

Problemet har homogena Dirichletvillkor (se ex. 3.1)

\Rightarrow Jag ansätter en sinusserie:

$$U(x, t) = \sum_{k=1}^{\infty} u_k(t) \cdot \sin(k\pi x)$$

Insättning i PDE ger: $\sum_1^{\infty} (u'_k + u_k \cdot k^2 \pi^2) \sin k\pi x = 0$

$$\Rightarrow u_k(t) = \beta_k e^{-k^2 \pi^2 t}$$

$$\Rightarrow u(x, t) = \sum_{k=1}^{\infty} \beta_k e^{-k^2 \pi^2 t} \cdot \sin(k\pi x)$$

$$u(x, 0) = \sum_1^{\infty} \beta_k \cdot \sin(k\pi x) = \sin \pi x + 2 \sin 3\pi x$$

$$\Rightarrow \beta_k = \begin{cases} 1 & \text{om } k = 1 \\ 2 & \text{om } k = 3 \\ 0 & \text{annars} \end{cases}$$

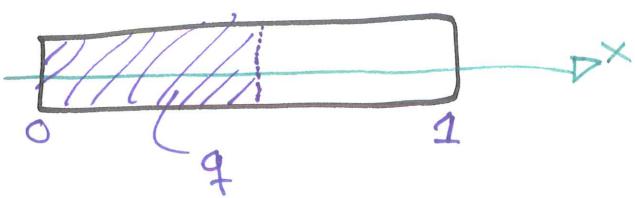
Vi behöver alltså bara två termer fr. $U(x, t)$.

$$U(x, t) = e^{-\pi^2 t} \cdot \sin \pi x + 2 e^{-9\pi^2 t} \cdot \sin 3\pi x$$

3.7

Ställ upp en matematisk modell och lös problemet

$$\text{PDE} \left\{ \begin{array}{l} U_t - U_{xx} = 0, \quad 0 < x < 1, \quad t > 0 \\ U_x(0,t) = U_x(1,t) = 0, \quad t > 0 \\ U(x,0) = q(\Theta(x) - \Theta(x-0.5)) \end{array} \right.$$



Vi har homogena Neumannvillkor \Rightarrow Ansätt en cosinusserie:

$$U(x,t) = \sum_{k=1}^{\infty} U_k(t) \cdot \cos(k\pi x) + \alpha_0$$

Insättning i PDE ger:

$$\sum_{k=1}^{\infty} (U_k + U_k \pi^2 k^2) \cos(k\pi x) = 0 \Rightarrow U_k(t) = \alpha_k e^{-\pi^2 k^2 t}$$

$$\Rightarrow U(x,t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cdot e^{-\pi^2 k^2 t} \cdot \cos(k\pi x)$$

Vi får α_0 & α_k mha BN.

$$\alpha_0 = \frac{1}{1} \int_0^1 U(x,0) dx = \int_0^{0.5} q dx = \boxed{\frac{q}{2}}$$

$$\alpha_k = 2 \int_0^{0.5} q \cdot \cos(k\pi x) dx = \frac{q \cdot 2}{k\pi} \left[\sin(k\pi x) \right]_0^{0.5} = \boxed{\frac{2q \sin(\frac{k\pi}{2})}{k\pi}}$$

svar: $U(x,t) = \frac{q}{2} + \sum_{k=1}^{\infty} \frac{2q}{k\pi} \sin\left(\frac{k\pi}{2}\right) e^{-k^2 \pi^2 t} \cdot \cos(k\pi x)$

3.8

Lös värmefördelningsproblem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & 0 < x < 1, t > 0 \\ u(x, 0) = x & 0 < x < 1 \end{cases}$$

a) Randvillkor: $u(0, t) = u(1, t) = 0$

Eftersom $\sin(0) = \sin(2\pi) = 0$ så ansätter jag en sinusserie som lösning i x-led.

$$u(x) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi x}{L}\right), \quad \beta_k = \frac{2}{L} \int_0^L u(x) \sin\left(\frac{k\pi x}{L}\right) dx$$

I vårt fall är $L = 1$.

Jag deriverar $u(x)$ termvis.

$$\frac{\partial u}{\partial x} = \sum_{k=1}^{\infty} (\beta_k^* (t) \cdot k\pi \cos(k\pi x))$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=1}^{\infty} (\beta_k \cdot k^2 \pi^2 \cdot \sin(k\pi x))$$

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} \beta_k^*(t) \sin(k\pi x)$$

Insättning i diffekvationen:

$$\sum_{k=1}^{\infty} ((\beta_k' + \beta_k k^2 \pi^2) \sin(k\pi x)) = 0$$

Entydigheten i f-serier ger att

$$\beta_k' + \beta_k k^2 \pi^2 = 0 \quad , \quad k = 1, 2, 3, \dots$$

$$\Rightarrow \boxed{\beta_k(t) = C_k e^{-k^2 \pi^2 t}}$$

$$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} C_k e^{-k^2 \pi^2 t} \cdot \sin(k\pi x)$$

$$u(x,0) = x = \sum_{k=1}^{\infty} C_k \cdot \sin(k\pi x)$$

$$C_k = \frac{2}{1} \int_0^1 x \cdot \sin\left(\frac{k\pi x}{1}\right) dx = 2 \left[-\frac{x \cos(k\pi x)}{k\pi} \right]_0^1$$

$$-2 \int_0^1 -\frac{\cos(k\pi x)}{k\pi} dx = 2 \frac{\cos(k\pi)}{k\pi} + 2 \frac{\sin(k\pi)}{k\pi} =$$

$$= \frac{2}{k\pi} (\cos(k\pi) + \sin(k\pi)) =$$

$$= \frac{-2}{k\pi} (-1)^k = \boxed{\frac{2}{k\pi} (-1)^{k+1}}$$

$$\Rightarrow U(x, t) = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{2}{k\pi} e^{-k^2\pi^2 t} \cdot \sin(k\pi x)$$

b) $U_x'(0, t) = U_x'(1, t) = 0$

Ansatz: cosinusserie.

$$U(x) = C_0 + \sum_{k=1}^{\infty} \alpha_k \cos\left(\frac{k\pi}{L} x\right) \quad , \quad L = 1$$

$$\frac{\delta U}{\delta x} = - \sum_{k=1}^{\infty} \alpha_k \sin(k\pi x) \cdot k\pi$$

$$\frac{\delta U}{\delta x^2} = - \sum_{k=1}^{\infty} \alpha_k k^2 \pi^2 \cos(k\pi x)$$

$$\frac{\delta U}{\delta t} = \sum_{k=1}^{\infty} \alpha_k' \cos(k\pi x)$$

$$\Rightarrow \sum_{k=1}^{\infty} (\alpha_k' + \alpha_k k^2 \pi^2) \cos(k\pi x) = 0$$

* har lösningarna ~~α_k'~~ $C_k e^{-k^2\pi^2 t}$, $k = 1, 2, 3 \dots$

$$\Rightarrow U(x, t) = \sum_{k=1}^{\infty} (C_k e^{-k^2\pi^2 t} \cdot \cos(k\pi x)) + C_0$$

$$U(x, 0) = X = C_0 + \sum_{k=1}^{\infty} C_k \cdot \cos(k\pi x)$$

fs: $C_0 = \frac{1}{1} \int_0^1 x dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2}$

$$c_k = \frac{2}{l} \int_0^l x \cdot \cos(k\pi x) dx =$$

$$= 2 \left[x \frac{\sin(k\pi x)}{k\pi} \right]_0^l - 2 \int_0^l \frac{\sin(k\pi x)}{k\pi} dx =$$

$$= \frac{2}{k\pi} (\sin(k\pi) + \cos(k\pi)) =$$

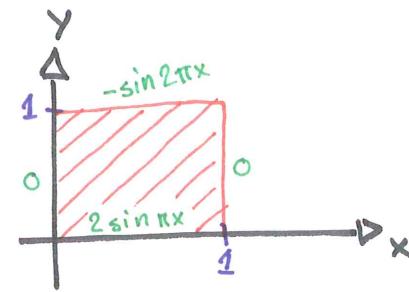
$$= \frac{2}{k\pi} (0 + (-1)^k) = \frac{2}{k\pi} (-1)^k$$



3.10

Lös randvärdesproblemet

$$\left\{ \begin{array}{l} U_{xx}'' + U_{yy}'' = 0, \quad 0 < x < 1, \quad 0 < y < 1 \\ U(x, 0) = 2 \sin(\pi x), \quad 0 < x < 1 \\ U(x, 1) = -\sin(2\pi x), \quad 0 < x < 1 \\ U(0, y) = U(1, y) = 0, \quad 0 < y < 1 \end{array} \right.$$



Vi ansätter: $U(x, y) = \sum_{k=1}^{\infty} U_k(y) \cdot \sin(k\pi x)$

Insättning i PDE:

$$\sum_{k=1}^{\infty} (U_k''(y) - U_k(y) k^2 \pi^2) \sin(k\pi x) = 0$$

$$\Rightarrow U_k(y) = A_k \cosh(k\pi y) + B_k \sinh(k\pi y)$$

TIPS!

$\cosh(0) = 1$
 $\sinh(0) = 0$

$$U(x, 0) = \sum_{k=1}^{\infty} (A_k \cdot 1 + B_k \cdot 0) \sin(k\pi x) = 2 \sin(\pi x)$$

$$\Rightarrow A_k = \begin{cases} 2 & \text{om } k = 1 \\ 0 & \text{annars} \end{cases}$$

$$U(x, 1) = \sum_{k=1}^{\infty} (A_k \cosh(k\pi) + B_k \sinh(k\pi)) \sin(k\pi x) = -\sin(2\pi x)$$

Vi betraktar tre fall:

$$\left\{ \begin{array}{l} k=1: 2 \cosh(\pi) + B_1 \sinh(\pi) = 0 \Rightarrow B_1 = -2 \frac{\cosh(\pi)}{\sinh(\pi)} \end{array} \right.$$

$$\left. \begin{array}{l} k=2: B_2 \sinh(2\pi) = -1 \Rightarrow B_2 = -\frac{1}{\sinh(2\pi)} \end{array} \right.$$

$$\left. \begin{array}{l} k > 2: B_k = 0 \end{array} \right.$$

$$\Rightarrow U(x, y) = \left(2 \cosh(\pi y) - 2 \frac{\cosh(\pi)}{\sinh(\pi)} \sinh(\pi y) \right) \sin(\pi x) - \frac{\sin(2\pi y)}{\sinh(2\pi)} \sin(2\pi x)$$

3.12

Lös svängningsproblem

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \quad t > 0 \\ u(0, t) = u(1, t) = 0 \quad t > 0 \\ u(x, 0) = 1 \quad 0 < x < 1 \\ u'_t(x, 0) = 0 \quad 0 < x < 1 \end{array} \right.$$

Jag ansätter en sinusserie.

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \cdot \sin \frac{k\pi x}{1}$$

Ansats

Derivering termvis ger:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{k=1}^{\infty} u''_k(t) \cdot \sin \frac{k\pi x}{1}, \quad \frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} u'_k(t) \cdot \sin(k\pi x)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=1}^{\infty} -u_k(t) \cdot k^2 \pi^2 \cdot \sin(k\pi x)$$

Insättning

$$\sum_{k=1}^{\infty} (\ddot{u} \cdot \sin(k\pi x) + 2\dot{u} \cdot \sin(k\pi x) + u k^2 \pi^2 \sin(k\pi x)) = 0$$

$$\Leftrightarrow \sum_{k=1}^{\infty} (\underbrace{\ddot{u} + 2\dot{u} + k^2 \pi^2 u}_{=0 \text{ (entydighet)}} \sin(k\pi x)) = 0$$

Vi löser diffekvationen

$$\ddot{U} + 2\dot{U} + k^2 \pi^2 U = 0$$

$$\text{Kar. pol: } r^2 + 2r + k^2 \pi^2 = 0$$

$$\Leftrightarrow r = -1 \pm \sqrt{1 - k^2 \pi^2} = -1 \pm i \sqrt{k^2 \pi^2 - 1}$$

$$\Rightarrow U_k(t) = A_k e^{(-1+i\sqrt{\pi^2})t} + B_k e^{(-1-i\sqrt{\pi^2})t} =$$

$$= e^{-t} (A_k \cos(\omega_k t) + B_k \sin(\omega_k t)) \quad (\omega_k = \sqrt{k^2 \pi^2 - 1})$$

$$\Rightarrow U(x,t) = \sum_{k=1}^{\infty} e^{-t} (A_k \cos(\omega_k t) + B_k \sin(\omega_k t)) \sin(k \pi x)$$

Vi tar fram A_k & B_k m.h.a. begynnelsevärden.

$$U(x,0) = 1 = \sum_{k=1}^{\infty} e^0 (A_k + 0) \sin(k \pi x)$$

$$\Leftrightarrow \sum_{k=1}^{\infty} A_k \sin(k \pi x) = 1$$

Utveckla ettan i en sinusserie

$$1 = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k \pi}{1} x\right), \beta_k = \frac{2}{1} \int_0^1 1 \cdot \sin(k \pi x) dx = -\frac{2}{k \pi} [\cos(k \pi x)]_0^1 =$$

$$= -\frac{2}{k \pi} (\cos(k \pi) - 1) = -\frac{2((-1)^k - 1)}{k \pi}$$

3.13

Lös värmelämningsproblem

$$\left\{ \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \quad t > 0 \right.$$

$$u(0, t) = 0, \quad u(1, t) = 1 \quad t > 0$$

$$u(x, 0) = 0 \quad 0 < x < 1$$

ej ej!

Vi börjar med att homogenisera randvillkoren.

En lösning är funktionen $u_{st}(x) = x$

$$\Rightarrow u_{st}(0) = 0, \quad u_{st}(1) = 1$$

Beror bara på x
och är stationär
i t (tid).

$$\frac{\partial^2 u_{st}}{\partial x^2} = 0$$

Vi bildar $v(x, t) = u(x, t) - u_{st}(x)$

$$\left\{ \begin{array}{l} v'_t - v''_{xx} = 0 \\ v(0, t) = v(1, t) = 0 \\ v(x, 0) = -x \end{array} \right.$$

Vi ansätter en sinusserie:

$$v(x, t) = \sum_{k=1}^{\infty} v_k(t) \cdot \sin(k\pi x)$$

Insättning i PDE ger:

$$\sum_{k=1}^{\infty} (v'_k + v_k \cdot k^2 \pi^2) \cdot \sin(k\pi x) = 0 \Rightarrow v_k = \beta_k e^{-k^2 \pi^2 t}$$

$$v(x, 0) = \sum_{k=1}^{\infty} \beta_k \cdot \sin(k\pi x) = -x \Rightarrow \sum_{k=1}^{\infty} \alpha_k \sin(k\pi x)$$

$$\Rightarrow \beta_k = \alpha_k = 2 \int_0^1 (-x) \sin(k\pi x) dx = \dots = \frac{2(-1)^k}{k\pi}$$

$$\Rightarrow v(x, t) = \sum_{k=1}^{\infty} \frac{2(-1)^k}{k\pi} \cdot e^{-k^2\pi^2 t} \cdot \sin(k\pi x)$$

$$u(x, t) = u_{\text{stat}} + v = x + \sum_{k=1}^{\infty} \frac{2(-1)^k}{k\pi} \cdot e^{-k^2\pi^2 t} \cdot \sin(k\pi x)$$

$$u(x, t) \rightarrow x \quad \text{da} \quad t \rightarrow \infty$$

3.14

Lös värmeförståndningsproblem

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = e^{-t} \cdot \sin(\pi x) \quad 0 < x < 1, \quad t > 0 \\ u(0, t) = u(1, t) = 0 \quad t > 0 \\ u(x, 0) = 0 \quad 0 < x < 1 \end{array} \right.$$

Homogena dirichletvillkor \Rightarrow sinusserie!

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \cdot \sin(k\pi x)$$

Jag deriverar $u(x, t)$ termvis och sätter in i ekvationen.

$$\sum_{k=1}^{\infty} (u'_k(t) + k^2 \pi^2 u_k(t)) \sin(k\pi x) = e^{-t} \cdot \sin(\pi x)$$

$$\Rightarrow k=1: u'_1 + \pi^2 u_1 = e^{-t} \Rightarrow u_1 = C_1 e^{-\pi^2 t} + \frac{1}{\pi^2 - 1} e^{-t}$$

Högerledet kan ses som en sinusserie

$$\sum_{k=1}^{\infty} v_k(t) \sin(\pi x k)$$

$$\text{där } v_k(t) = \begin{cases} e^{-t} & \text{då } k=1 \\ 0 & \text{då } k \geq 2 \end{cases}$$

$$k \geq 2 \Rightarrow u'_k + k^2 \pi^2 u_k = 0 \Rightarrow u_k = C_k e^{-k^2 \pi^2 t}$$

$$u(x, 0) = 0 \Rightarrow u_k(0) = 0 \quad \text{för alla } t.$$

$$\Rightarrow C_1 = -\frac{1}{\pi^2 - 1}, \quad C_k = 0, \quad k=2,3,\dots$$

$$u(x, t) = \frac{1}{\pi^2 - 1} (e^{-t} - e^{-\pi^2 t}) \sin(\pi x)$$

3.15

Lös värmeförståndningsproblem

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 1 \quad 0 < x < 1, t > 0 \\ u(0, t) = u(1, t) = 0 \quad t > 0 \\ u(x, 0) = 0 \quad 0 < x < 1 \end{array} \right.$$

Jag ansätter en sinusserie.

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(k\pi x)$$

$$\frac{\partial u}{\partial t} = \sum u'_k(t) \sin(k\pi x), \quad \frac{\partial^2 u}{\partial x^2} = \sum_{k=1}^{\infty} -u_k(t) k^2 \pi^2 \sin(k\pi x)$$

Insättning i diffekvationen:

$$\sum_{k=1}^{\infty} (u'_k(t) + u_k(t) k^2 \pi^2) \sin(k\pi x) = 1$$

Jag utvecklar högerledet i en sinusserie

$$1 = \sum_{k=1}^{\infty} b_k \sin(k\pi x), \quad b_k = \frac{2}{1} \int_0^1 1 \cdot \sin(k\pi x) dx = -\frac{2}{k\pi} [\cos(k\pi x)]_0^1 =$$

\uparrow
Tas ut.

$$= -\frac{2}{k\pi} (\cos(k\pi) - 1) = \frac{2}{\pi} \cdot \frac{1 - (-1)^k}{k}$$

$$\Rightarrow u'_k(t) + k^2 \pi^2 u_k(t) = \frac{2}{\pi} \cdot \frac{1 - (-1)^k}{k}$$

Detta är första gradens linjära diffekvation.



Vi använder en integrerande faktor:

$$\Rightarrow u_k \cdot e^{k^2 \pi^2 t} + k^2 \pi^2 \cdot e^{k^2 \pi^2 t} \cdot u_k = \frac{2}{\pi} \cdot \frac{1 - (-1)^k}{k}$$

$$\Leftrightarrow \frac{d}{dt} (u_k \cdot e^{k^2 \pi^2 t}) = e^{k^2 \pi^2 t} \cdot \frac{2}{\pi} \cdot \frac{1 - (-1)^k}{k}$$

$$\Rightarrow u_k \cdot e^{k^2 \pi^2 t} = \frac{e^{k^2 \pi^2 t}}{k^2 \pi^2} \cdot \frac{2}{\pi} \cdot \frac{1 - (-1)^k}{k} + A$$

$$u_k(0) = 0 \Rightarrow A = -\frac{2(1 - (-1)^k)}{\pi^3 \cdot k^3}$$

$$\Rightarrow u_k = \frac{2(1 - e^{-k^2 \pi^2 t})(1 - (-1)^k)}{\pi^3 \cdot k^3}$$

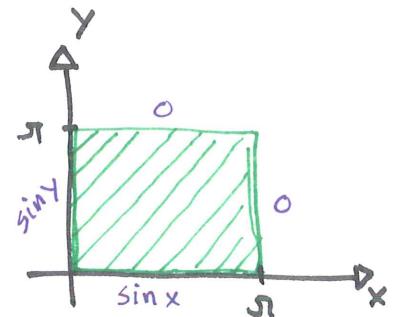
SVAR

$$u(x,t) = \sum_{k=1}^{\infty} \frac{2(1 - e^{-k^2 \pi^2 t})(1 - (-1)^k)}{\pi^3 k^3} \sin(k \pi x)$$

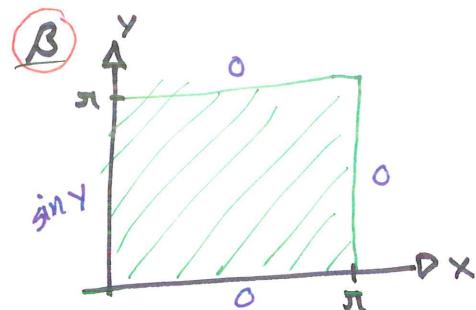
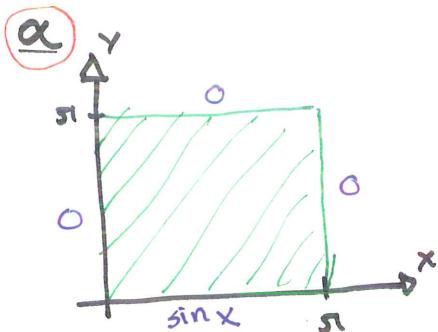
$$\lim_{t \rightarrow \infty} u(x,t) = \sum_{k=1}^{\infty} \frac{2(1 - (-1)^k)}{\pi^3 k^3} \sin(k \pi x)$$

3.19 Lös det stationära värmefördelningsproblem.

$$\left\{ \begin{array}{l} U_{xx}'' + U_{yy}'' = 0, \quad 0 < x < \pi, \quad 0 < y < \pi \\ U(x, 0) = \sin x, \quad U(x, \pi) = 0 \quad 0 < x < \pi \Rightarrow \\ U(0, y) = \sin y, \quad U(\pi, y) = 0 \quad 0 < y < \pi \end{array} \right.$$



Vi delar upp problemet i två delar α och β .



$U(x, y)$ ges nu av en superposition av u_α och u_β .

$\alpha :$

$$\text{Ansats: } u_\alpha(x, y) = \sum_{k=1}^{\infty} u_k^\alpha(y) \sin kx$$

$$\text{PDE: } \sum_{k=1}^{\infty} (u_k^{*\alpha}(y) - k^2 u_k^\alpha(y)) \sin kx = 0$$

$$\Rightarrow u_k^\alpha = A_k e^{ky} + B_k e^{-ky} = A_k \cosh(ky) + B_k \sinh(ky)$$

$$\text{Vi har } u_\alpha(x, y) = \sum_{k=1}^{\infty} (A_k \cosh(ky) + B_k \sinh(ky)) \sin(kx)$$

$$u_\alpha(x, 0) = \sum_{k=1}^{\infty} (A_k + 0) \sin(kx) = \sin x \Rightarrow A_k = \begin{cases} 1 & \text{då } k=1 \\ 0 & \text{då } k>1 \end{cases}$$

$$u_\alpha(x, \pi) = \sum_{k=1}^{\infty} (A_k \cosh(k\pi) + B_k \sinh(k\pi)) \sin(kx) = 0$$

$$\Rightarrow B_k = \begin{cases} -\frac{\cos(k\pi)}{\sinh(k\pi)}, & k=1 \\ 0, & k>1 \end{cases}$$

Vi får bara bidrag då $k=1$
Vi behöver inga sommar.

$$\Rightarrow u_\alpha(x, y) = (\cosh(y) - \frac{\cosh(\pi)}{\sinh(\pi)} \sinh(y)) \sin(x)$$

$\beta:$

Eftersom vi har symmetri i x- och y-led ser vi direkt att

$$U_p(x,y) = \left(\cosh(x) - \frac{\cosh(\pi)}{\sinh(\pi)} \sinh(x) \right) \sin y$$

Vi vet att $u(x,y)$ ges av superpositionen

$$u(x,y) = U_x(x,y) + U_p(x,y) =$$

$$= \left(\cosh(y) - \frac{\cosh(\pi)}{\sinh(\pi)} \sinh(y) \right) \sin(x) +$$

SVAR:

$$+ \left(\cosh(x) - \frac{\cosh(\pi)}{\sinh(\pi)} \sinh(x) \right) \sin(y)$$

3.21



Situationen har varat så länge att jämvikt upphållt. Vid $t=0$ avbryts upphettningen, varefter änden $x=0$ ges temp T_0 .

Beräkna avkylningsförloppet.

Vi bestämmer först begynnelsetermineraturen.

$$\begin{cases} \tilde{u}_{xx}(x) = 0 & \text{(inget } t\text{-beroende längre)} \\ \tilde{u}(0) = T_i \\ \tilde{u}(L) = T_0 \end{cases}$$

$$\Rightarrow \tilde{u}(x) = Ax + B$$

$$\tilde{u}(0) = T_i \Rightarrow B = T_i$$

$$\tilde{u}(L) = T_0 \Rightarrow T_0 = A \cdot L + T_i \Leftrightarrow A = \frac{T_0 - T_i}{L}$$

$$\Rightarrow \tilde{u}(x) = \frac{T_0 - T_i}{L} x + T_i = u(x, 0)$$

Vi har nu funnit BV för vårt nya system.

$$\begin{cases} U_t - a^2 U_{xx}'' = 0 & , 0 < x < L, t > 0 \\ U(0, t) = T_0, U(L, t) = T_0 & t > 0 \\ U(x, 0) = T_i + \frac{T_0 - T_i}{L} x & , 0 < x < L \end{cases}$$

PDE

RV

BV

Vi homogeniseras genom att införa $V = U - T_0$.

Nya RV: $V(0, t) = V(L, t) = 0$

Nya BV: $V(x, 0) = T_i + \frac{T_0 - T_i}{L} x - T_0$

Nu har vi BV och RV som vi kan jobba med.

$$\text{Ansats: } v(x,t) = \sum_{k=1}^{\infty} v_k(t) \sin\left(\frac{k\pi x}{L}\right)$$

(PDE): $\sum_{k=1}^{\infty} (v'_k + \alpha \left(\frac{k\pi}{L}\right)^2 \cdot v_k) \sin\left(\frac{k\pi x}{L}\right) = 0$

$$\Rightarrow v_k(t) = C_k \cdot e^{-\alpha \left(\frac{k\pi}{L}\right)^2 t}$$

$$\Rightarrow v(x,t) = \sum_{k=1}^{\infty} C_k \cdot e^{-\alpha \left(\frac{k\pi}{L}\right)^2 t} \cdot \sin\left(\frac{k\pi x}{L}\right)$$

(BV): $v(x,0) = \sum_{k=1}^{\infty} C_k \cdot \sin\left(\frac{k\pi x}{L}\right) = T_1 - T_0 + \frac{T_0 - T_1}{L}$

Vi får C_k genom att ta fram koeficienterna B_k till den sinusserie vi kan omforma till.

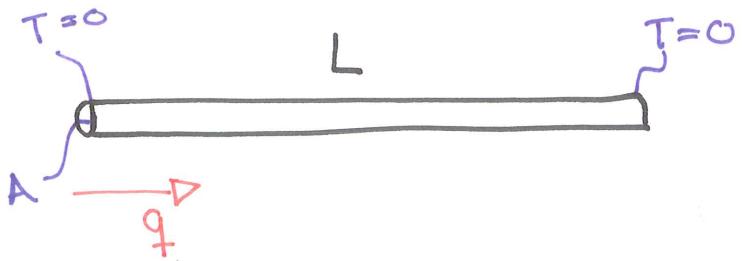
$$C_k = B_k = \frac{2}{L} \int_0^L \left(T_1 - T_0 + \frac{T_0 - T_1}{L} \right) \sin\left(\frac{k\pi x}{L}\right) dx =$$

$$= \frac{2}{L} \left(T_1 - T_0 + \frac{T_0 - T_1}{L} \right) \left(-\frac{k\pi}{L}\right) \left[\cos\left(\frac{k\pi x}{L}\right) \right]_0^L =$$

$$= -\frac{2k\pi}{L^2} \left(T_1 - T_0 + \frac{T_0 - T_1}{L} \right) ((-1)^k - 1)$$

FAN.

3.22



$$u(x, 0) = 0, \quad 0 < x < L, \quad t = 0$$

$$u(0, t) = u(L, t) = 0$$

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = \frac{\alpha q}{\lambda A}, \quad 0 < x < L, \quad t > 0$$

Värmededning: FS, $k = \frac{q}{A}$

Bestäm temperaturförlöppet

Ansats

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \cdot \sin\left(\frac{k\pi x}{L}\right)$$

Vi sätter in u'_k och u''_k (Termvis deriverat)

$$\Rightarrow \sum_{k=1}^{\infty} (u'_k + \alpha \left(\frac{k\pi}{L}\right)^2 \cdot u_k) \sin\left(\frac{k\pi x}{L}\right) = \frac{\alpha q}{\lambda A}$$

Jag utvecklar högerledet i en sinusserie

$$\frac{\alpha q}{\lambda A} = \sum_{k=1}^{\infty} b_k \cdot \sin\left(\frac{k\pi x}{L}\right), \quad b_k = \frac{2}{L} \int_0^L \frac{\alpha q}{\lambda A} \cdot \sin\left(\frac{k\pi x}{L}\right) dx =$$

Enkel diff.-
ekvation!

$$= \frac{2}{L} \cdot \frac{\alpha q}{\lambda A} \cdot \frac{-L}{k\pi} (\cos(k\pi) - \cos(0))$$

$$\Rightarrow u'_k + \alpha \left(\frac{k\pi}{L}\right)^2 \cdot u_k = \frac{2\alpha q}{k\pi \lambda A} (1 - (-1)^k)$$

Vi använder en integrerande faktor (se endim)

$$\Rightarrow \frac{d}{dt} \left(U_k(t) \cdot e^{a \frac{k^2 \pi^2}{L^2} t} \right) = e^{a \frac{k^2 \pi^2}{L^2} t} \cdot \frac{2aq}{k\pi\lambda A} (1 - (-1)^k)$$

Integrering och begynnelsevillkor ger:

$$U_k(t) \cdot e^{a \frac{k^2 \pi^2}{L^2} t} = \frac{L^2}{ak^3 \pi^2} \cdot e^{a \frac{k^2 \pi^2}{L^2} t} \cdot \frac{2aq}{k\pi\lambda A} (1 - (-1)^k) - \frac{L^2}{ak^2 \pi^2} \cdot \frac{2aq}{\lambda A} \cdot \frac{(1 - (-1)^k)}{k\pi}$$

$$= \boxed{U_k(t) = \frac{2L^2 q}{\lambda A \pi^3} \cdot \frac{1 - (-1)^k}{k^3} (1 - e^{-a \frac{k^2 \pi^2}{L^2} t})}$$

SVAR

$$\Rightarrow \boxed{U(x,t) = \sum_{k=1}^{\infty} \frac{2L^2 q}{\lambda A \pi^3} \cdot \frac{1 - (-1)^k}{k^3} (1 - e^{-a \frac{k^2 \pi^2}{L^2} t}) \sin \left(\frac{k\pi x}{L} \right)}$$

3.24

a) Bestäm den stationära temperaturfördelningen.

$$\text{PDE: } \begin{cases} U_{xx}'' + U_{yy}'' = 0, & 0 < x < b, 0 < y < h \\ U(0,y) = U(b,y) = 0 \end{cases}$$

$$\text{RV: } U(0,y) = U(b,y) = 0$$

$$\text{RV: } U(x,0) = 0, U(x,h) = g(x) = \begin{cases} \frac{2T_0x}{b}, & 0 < x < b/2 \\ \frac{2T_0}{b}(b-x), & b/2 < x < b \end{cases}$$

Eftersom vi har homogena Dirichletvillkor i x-led så ansätter vi lösningen som en sinusserie i x-led.

$$U(x,y) = \sum_{k=1}^{\infty} u_k(y) \cdot \sin\left(\frac{k\pi x}{b}\right)$$

$$\text{PDE: } \sum_{k=1}^{\infty} \left(u_k'' - \frac{k^2\pi^2}{b^2} u_k \right) \sin\left(\frac{k\pi x}{b}\right) = 0$$

$$\Rightarrow u_k(y) = a_k \cdot \cosh\left(\frac{k\pi y}{b}\right) + b_k \cdot \sinh\left(\frac{k\pi y}{b}\right)$$

$$\text{RV: } U(x,0) = \sum a_k \cdot \sin\left(\frac{k\pi x}{b}\right) = 0 \Rightarrow a_k = 0$$

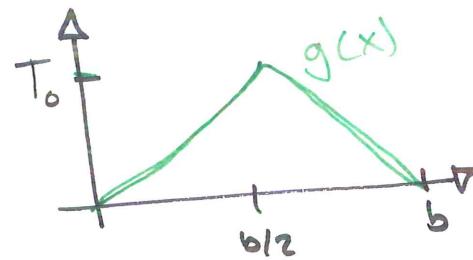
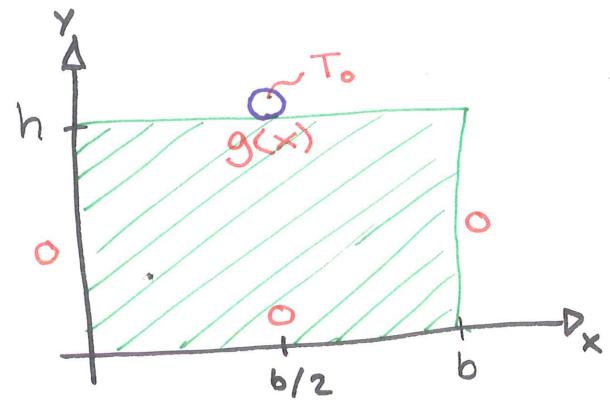
sinusutveckling
av $g(x)$.

$$U(x,h) = \sum b_k \cdot \sinh\left(\frac{k\pi h}{b}\right) \sin\left(\frac{k\pi x}{b}\right) = g(x) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi x}{b}\right)$$

$$\beta_k = \frac{2}{b} \int_0^b g(x) \cdot \sin\left(\frac{k\pi x}{b}\right) dx = \frac{2}{b} \cdot \frac{2T_0}{1} \left(\int_0^{b/2} \frac{x}{b} \sin\frac{k\pi x}{b} dx + \int_{b/2}^b \left(1 - \frac{x}{b}\right) \sin\frac{k\pi x}{b} dx \right)$$

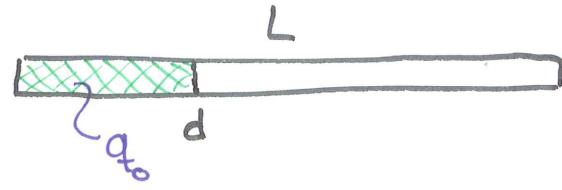
SVAR

$$\Rightarrow U(x,y) = \sum \dots \text{ se facit.}$$



3.25

Lös diffusionsproblem.



PDE $\left\{ \begin{array}{l} U_t' - D U_{xx}'' = -c \cdot u \quad , \quad 0 < x < L, \quad t > 0 \\ U_x(0, t) = U_x(L, t) = 0 \quad , \quad t > 0 \end{array} \right.$

RV $\left\{ \begin{array}{l} U_x(0, t) = U_x(L, t) = 0 \quad , \quad t > 0 \end{array} \right.$

BV $\left\{ \begin{array}{l} U(x, 0) = \begin{cases} q_0 & , \quad 0 < x < d \\ 0 & , \quad x > d \end{cases} \quad , \quad 0 < x < L \end{array} \right.$

Vi har homogena Neumannvillkor

\Rightarrow Ansats: $U(x, t) = \sum_{k=1}^{\infty} U_k(t) \cdot \cos\left(\frac{k\pi x}{L}\right) + C_0$

PDE: $\sum_{k=1}^{\infty} \left(U_k' + D \frac{k^2 \pi^2}{L^2} U_k \right) \cos\left(\frac{k\pi x}{L}\right) = -c \cdot u = \sum_{k=1}^{\infty} -c U_k \cdot \cos\left(\frac{k\pi x}{L}\right)$

$$= -c U_k$$

$\Rightarrow U_k' + \left(D \frac{k^2 \pi^2}{L^2} + c \right) U_k = 0 \Rightarrow U_k = C_k \cdot e^{-\left(D \frac{k^2 \pi^2}{L^2} + c \right) t}$

$$U(x, t) = \sum_{k=1}^{\infty} C_k \cdot e^{-\left(D \frac{k^2 \pi^2}{L^2} + c \right) t} \cdot \cos\left(\frac{k\pi x}{L}\right)$$

$$U(x, 0) = \sum_{k=1}^{\infty} C_k \cdot \cos\left(\frac{k\pi x}{L}\right) = q_0 (\Theta(x) - \Theta(x-d))$$

Vi skriver HL som en cosinusserie vars koefficienter $\beta_k = C_k$.

$$\Rightarrow C_k = \beta_k = \frac{2}{L} \int_0^d q_0 \cos\left(\frac{k\pi x}{L}\right) dx = \frac{2 q_0 L}{L k \pi} \left[\sin\left(\frac{k\pi x}{L}\right) \right]_0^d =$$

$$= \frac{2 q_0}{k \pi} \sin\left(\frac{k\pi d}{L}\right)$$

Vi har alltså:

$$U(x,t) = \sum_{k=1}^{\infty} \frac{2q_0}{k\pi} \sin\left(\frac{k\pi d}{L}\right) \cdot e^{-(D^2 \frac{k^2 \pi^2}{L^2} + c)t} \cdot \cos\left(\frac{k\pi x}{L}\right) + C_0$$

Vi ter fram C_0 .

FS: $C_0 = \frac{1}{L} \int_0^L U(x,0) dx = \frac{1}{L} \int_0^L q_0 dx = \frac{q_0 d}{L}$

Denna har
vi inte än.

Svar:

$$U(x,t) = \sum_{k=1}^{\infty} \frac{2q_0}{k\pi} \sin\left(\frac{k\pi d}{L}\right) \cdot e^{-(D^2 \frac{k^2 \pi^2}{L^2} + c)t} \cdot \cos\left(\frac{k\pi x}{L}\right) + \frac{q_0 d}{L}$$

3.28

a) $\begin{cases} U_{tt}'' - c^2 U_{xx}'' = -g \\ U(0, t) = U(L, t) = 0 \\ U(x, 0) = 0 \\ U_t(x, 0) \neq 0 \end{cases}$

b) Lös svängningsproblemet

Kolla facit, herregud, jag tänker inte ens försöka.

3.29

Diffusion

$$\begin{cases} \dot{U}_t - D U_{xx}'' = 0 & , 0 < x < L, t > 0 \\ U(0, t) = q_0, \dot{U}_x(L, t) = 0 & , t > 0 \\ U(x, 0) = 0 & , 0 < x < L \end{cases}$$

Vi har inhomogena randvillkor och måste homogenisera.

Vi sätter $v(x, t) = u(x, t) - q_0$.

$$\Rightarrow \dot{v}_t = \dot{u}_t, v_{xx}'' = U_{xx}''$$

$$\Rightarrow \begin{cases} \dot{v}_t - D v_{xx}'' = 0 \\ v(0, t) = 0, \dot{v}_x(L, t) = 0 \\ v(x, 0) = q_0 \end{cases}$$

Homogen!

Vi identifierar Sturm-Liouville-operatorn

$$A = -\frac{\partial^2}{\partial x^2}, D_A = \left\{ u \in C^2[0, L]; u(0) = \dot{u}_x(L) = 0 \right\}$$

Egenvärden och egenvektorer

$\lambda_k \neq 0, \lambda_k < 0$ ty A är positivt semidefinit.

~~$$A \varphi_k = \lambda_k \varphi_k \Leftrightarrow \varphi_k'' = \lambda_k \varphi_k \Rightarrow \varphi(x) = a \cos(\sqrt{\lambda_k} x) + b \sin(\sqrt{\lambda_k} x)$$~~

$$\varphi(0) = a \cdot \cos(0) + b \sin(0) = a = 0$$

$$\Rightarrow \varphi(x) = b \cdot \sin(\sqrt{\lambda_k} x)$$

$$\varphi'(x) = b \sqrt{\lambda_k} \cos(\sqrt{\lambda_k} x)$$

$$\varphi'(L) = b \sqrt{\lambda_k} \cos(\sqrt{\lambda_k} L) = 0 \Leftrightarrow \sqrt{\lambda_k} \cdot L = \frac{\pi}{2} + \pi k$$

Egenvärden.

$$\Rightarrow \lambda_k = \frac{\pi^2}{L^2} \left(\frac{1}{2} + k \right)^2, \quad k \in \mathbb{Z}$$

Eigenfunktioner

$$\varphi_k(x) = \sin\left(\frac{\pi}{L} \left(\frac{1}{2} + k\right)x\right)$$

Funktionerna φ_k är en ortogonal bas i $L_2[0, L]$.

Ansats: $v(x, t) = \sum_{k=0}^{\infty} v_k(t) \cdot \varphi_k(x)$

Vi sätter in ansatsen i diff.ekvationen $v_t' + DAv = 0$.

Derivation termvis och $A\varphi_k = \lambda_k \cdot \varphi_k$ ger:

$$\sum_{k=0}^{\infty} (v_k' + D\lambda_k \cdot v_k) \varphi_k(x) = 0$$

Entydighet ger:

$$v_k' + D\lambda_k \cdot v_k = 0 \Rightarrow v_k(t) = C_k \cdot e^{-D\lambda_k t}$$

Vi har alltså:

$$v(x, t) = \sum_{k=0}^{\infty} C_k \cdot e^{-D\lambda_k t} \cdot \varphi_k(x)$$

Hur får vi fram C_k ?

Begynnelsevillkor ger:

$$V(x, 0) = \sum_{k=0}^{\infty} c_k \cdot \varphi_k(x) = -q_0$$

Vi utvecklar $-q_0$ i basen $\{\varphi_k\}_{k=0}^{\infty}$

$$\Rightarrow c_k = \frac{(\varphi_k | V(x, 0))}{(\varphi_k | \varphi_k)} = \frac{(\varphi_k | -q_0)}{(\varphi_k | \varphi_k)}.$$

$$\bullet (\varphi_k | -q_0) = \int_0^L \sin(\sqrt{\lambda_k} x) (-q_0) dx = -\frac{q_0}{\sqrt{\lambda_k}}$$

$$\bullet (\varphi_k | \varphi_k) = \int_0^L \sin^2(\sqrt{\lambda_k} x) dx = \dots = \frac{L}{2}$$

$$\Rightarrow c_k = \frac{-q_0}{\frac{L}{2}} = -\frac{2q_0}{L \cdot \sqrt{\lambda_k}}$$

$$\Rightarrow V(x, t) = \sum_{k=0}^{\infty} -\frac{2q_0}{\sqrt{\lambda_k} L} \cdot e^{-D\lambda_k t} \cdot \varphi_k(x)$$

SVAR

$$\Rightarrow U(x, t) = q_0 - \frac{2q_0}{L} \sum_{k=0}^{\infty} \frac{1}{\sqrt{\lambda_k}} e^{-D\lambda_k t} \cdot \varphi_k(x)$$

3.35

Bestäm modul och egenfrekvenser för ljudvågor i det exponentialhorn som uppkommer då

$$Y = k e^{ax}, \quad 0 \leq x \leq L$$

roterar kring x-axeln. Hornet antas vara öppet i bågge ändar.

Vägekvationen

$$\left\{ \begin{array}{l} U_{tt}'' - v \cdot \frac{1}{A(x)} \cdot \frac{d}{dx} (A(x) \cdot U_x') = 0 \\ U(0, t) = U(L, t) = 0 \quad t > 0 \end{array} \right.$$

Vi vill beräkna egenvärden λ_k och egenfunktioner φ_k till operatorn:

$$A = -e^{-2ax} \frac{\delta}{\delta x} \left(e^{2ax} \frac{\delta}{\delta x} \right)$$

$$D_A = \left\{ v \in C^2[0, L] \mid v(0) = v(L) = 0 \right\}$$

3.39

Lös egenvärdesproblemet.

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \frac{\delta}{\delta x} \left((1-x^2) \frac{\delta u}{\delta x} \right) = 0, \quad -1 < x < 1, \quad t > 0 \\ u(x,0) = x^2 \\ u \text{ är begränsad} \end{array} \right.$$

Vi använder att:Operatorn $Au = -\frac{d}{dx} \left((1-x^2) \frac{du}{dx} \right)$, $-1 < x < 1$

har en ortogonal bas av egenfunktioner $\{P_k\}_{k=0}^{\infty}$
 i $L_2[-1,1]$, där P_k är ett polynom av grad k
 (Legendrepolynom).

FS: $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

Enligt sats 5.4: $\lambda_k = k(k+1)$.

Vi kan skriva om värmeförädlingsproblem:

PDE $\left\{ \begin{array}{l} u_t + Au = 0, \quad -1 < x < 1, \quad t > 0 \\ u(x,0) = x^2 \\ u \text{ är begränsad} \end{array} \right.$

Operatorn A är en singulär SL-operator och
 egenfunktionerna P_k är en ortogonal bas i $L_2(-1,1)$.

Ansats: $u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cdot P_k(x)$

Vi sätter in i PDE

$$\sum_{k=0}^{\infty} (U'_k + \lambda_k \cdot U_k) P_k = 0 \Rightarrow U_k = C_k e^{-\lambda_k t}$$

BV: $U(x, 0) = \sum_{k=0}^{\infty} C_k P_k(x) = x^2$

Eftersom x^2 är ett andragradspolynom är
såklart $U(x, 0)$ också ett andragradspolynom.

$$\Rightarrow U(x, 0) = x^2 = \frac{2}{3} P_2 - \frac{1}{3} P_0$$

$$\Rightarrow C_k = \begin{cases} -1/3, & k=0 \\ 2/3, & k=2 \\ 0, & \text{annars} \end{cases}$$

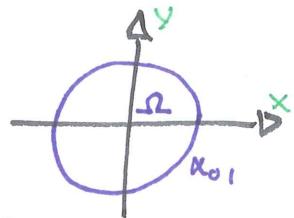
SVAR

$$\Rightarrow U(x, t) = -\frac{1}{3} + \frac{2}{3} e^{-6t} (3x^2 - 1) \cdot \frac{1}{2}$$

$$\lambda_0 = 0, \lambda_1 = 2, \lambda_2 = 6$$

3.4 | Lös svängningsproblem

$$u(r, \theta) = J_0(r)$$



$$U_{tt}'' - c^2 \Delta u = 0, \quad t > 0, \quad 0 < r < \alpha_0,$$

$$U(\alpha_0, t) = 0, \quad t > 0$$

$$U(r, 0) = J_0(r), \quad 0 < r < \alpha_0,$$

$$U_t'(r, 0) = 0, \quad 0 < r < \alpha_0,$$

U är begränsad

Vi har alltså operatorn

$$A = -\Delta = -\frac{1}{r} \left(\frac{d}{dr} \left(r \frac{d}{dr} \right) \right)$$

$$D_A = \{v \in C^1[0, \alpha_0] \mid v(\alpha_0) = 0, v \text{ begränsad}\}$$

Vi börjar med att bestämma egenfunktioner och egenvärden till A .

$$A \varphi_k = \lambda_k \varphi_k \iff -\frac{1}{r} (r \varphi'_k)' = \lambda_k \varphi_k$$

$$\iff \varphi_k'' + \frac{1}{r} \varphi_k' + \lambda_k \varphi_k = 0 \quad *$$

* är Bessels diff.ekvation med $v=0$ och har den allmänna lösningen:

$$\varphi_k(r) = a J_0(\sqrt{\lambda_k} r) + b Y_0(\sqrt{\lambda_k} r)$$

$b = 0$ eftersom
 v är begränsad!

$$\varphi(\alpha_{01}) = 0$$

$$\Rightarrow \sqrt{\lambda_k} \cdot \alpha_{01} = \alpha_{0k}, \quad k \in \mathbb{Z}$$

$$\Leftrightarrow \lambda_k = \left(\frac{\alpha_{0k}}{\alpha_{01}} \right)^2, \quad \varphi_k(r) = J_0\left(\frac{\alpha_{0k}}{\alpha_{01}} r \right), \quad k \in \mathbb{Z}$$

Vi ansätter som vanligt

$$U(r,t) = \sum_{k=1}^{\infty} u_k(t) \cdot \varphi_k(r)$$

Och sätter in i PDE ger

$$u''_k(t) + c^2 \cdot \lambda_k u_k(t) = 0$$

$$\Leftrightarrow u_k(t) = a_k \cdot \cos\left(c \frac{\alpha_{0k}}{\alpha_{01}} t\right) + b_k \cdot \sin\left(c \frac{\alpha_{0k}}{\alpha_{01}} t\right)$$

$$\Rightarrow U(r,t) = \sum_{k=1}^{\infty} \left(a_k \cdot \cos\left(c \frac{\alpha_{0k}}{\alpha_{01}} t\right) + b_k \sin\left(c \frac{\alpha_{0k}}{\alpha_{01}} t\right) J_0\left(\frac{\alpha_{0k}}{\alpha_{01}} r\right) \right)$$

$$u'_k(r,0) = 0 \Rightarrow b_k = 0$$

$$U(r,0) = J_0(r) = \sum_{k=1}^{\infty} a_k J_0\left(\frac{\alpha_{0k}}{\alpha_{01}} r\right) \Rightarrow \begin{cases} a_1 = 1 \\ a_k = 0, \quad k = 2, 3, \dots \end{cases}$$

SVAR

$$\Rightarrow U(r,t) = \cos(c \cdot t) \cdot J_0(r)$$

3.42

Bestäm kritiska diametern för en lång stav.

$$\text{PDE: } U_t - a \Delta u = c u$$

$$\text{RV: } U(R, \theta, z) = 0$$



$$U(r, \theta, z)$$

Ansats:
$$U(r, \theta, z) = \sum_{k=1}^{\infty} U_k(t) \cdot \varphi_k(r)$$

Vi har alltså operatorn

$$A = -a \Delta = -\frac{a}{r} \left(\frac{d}{dr} \left(r \cdot \frac{du}{dr} \right) \right), D_A = \{ U \in C^2(0, R) \mid U(R) = 0, U \text{ begr} \}$$

där vi använt att Δ är laplaceoperatorn i cylindriska koordinater och att vi har symmetri i θ - & z -led.

Vi vill alltså nu finna egenvärden och egenvektorer till:

$$A \varphi_k = \lambda_k \varphi_k$$

$$\Leftrightarrow -\frac{1}{r} (r \varphi'_k)' = \lambda_k \varphi_k$$

$$\Leftrightarrow -a \varphi''_k - \frac{a}{r} \varphi'_k - \lambda_k \varphi_k = 0$$

$$\Leftrightarrow \varphi''_k + \frac{1}{r} \varphi'_k + \frac{\lambda_k}{a} \varphi_k = 0$$

Bessels ekvation!

Eftersom φ_k är begränsad har vi:

$$\varphi_k = C \cdot J_v \left(\sqrt{\frac{\lambda_k}{a}} r \right) \Rightarrow r = \alpha_{v1} \sqrt{\frac{a}{\lambda_k}}$$

$$\text{PDE: } U_t' + Au = cu$$

Vi söker den stationära lösningen då $t \rightarrow \infty$

$$\Rightarrow Au = cu$$

Detta betyder att ett möjligt egenvärde är

$$\lambda_k = c \Rightarrow d = 2r = 2\alpha_{01} \sqrt{\frac{a}{c}}$$

svar: $d = 2\alpha_{01} \sqrt{\frac{a}{c}}$

3.45

Bestäm temp. i en sfärisk kula efter 10 min.

$$R = 0,2 \text{ m}, T_0 = 100^\circ\text{C}, U(R, t) = 0$$

PDE: $U_t - \alpha \Delta u = 0, 0 < r < R, t > 0$

RV: $U(R, t) = 0, t > 0, u \text{ begr.}$

BV: $U(r, 0) = T_0, 0 < r < R$

Vi har operatorn $A = -\alpha \Delta, D_A = \{v \in C^2[0, R] \mid v(R, t) = 0, v(r, 0) = T_0\}$

Egenvärden & egenfunktioner.

$$A \varphi_k = \lambda_k \varphi_k \Leftrightarrow -\frac{\alpha}{r^2} (r^2 \varphi'_k)' = \lambda_k \varphi_k$$

$$\Leftrightarrow -\frac{1}{r^2} r \varphi''_k - \frac{1}{r} \varphi'_k = \lambda_k \varphi_k$$

$$\Leftrightarrow \varphi''_k + \frac{2}{r} \varphi'_k + \frac{\lambda_k}{r} \varphi_k = 0 \quad \text{Bessel ekvation!}$$

$$\Rightarrow \varphi_k(r) = a J_0(\sqrt{\lambda_k} r) + b Y_0(\sqrt{\lambda_k} r)$$

$$\varphi_k(R) = 0 \Rightarrow J_0(\sqrt{\lambda_k} R) = 0 \Leftrightarrow \frac{\sin(\sqrt{\lambda_k} R)}{\sqrt{\lambda_k} R} = 0$$

$$\Rightarrow R \sqrt{\lambda_k} = \pi k \Leftrightarrow \lambda_k = \left(\frac{\pi k}{R}\right)^2 \text{ och } \varphi_k(r) = \frac{\sin\left(\frac{\pi k}{R} r\right)}{r}$$

u är begränsad

Vi ansätter (i enlighet med det förbannade "arbetsschemat"):

$$U(r, t) = \sum_{k=1}^{\infty} U_k(t) \varphi_k(r)$$

Insättning i PDE
ger som vanligt:

$$U_k(t) = C_k \cdot e^{-\alpha \lambda_k t}$$

Vi vill nu bestämma c_k och gör detta mha BV.

$$BV \Rightarrow u(r, 0) = \sum_{k=1}^{\infty} c_k(t) \varphi_k(r) = T_0 = 100$$

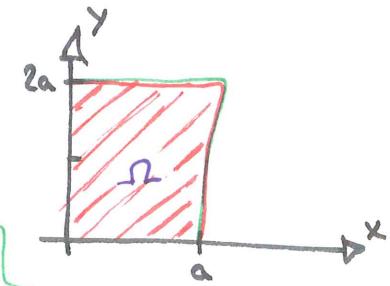
c_k ges av projektionen av T_0 på $\varphi_k(r)$

$$c_k = \frac{(T_0 | \varphi_k)}{(\varphi_k | \varphi_k)} = \frac{\int_0^{0,2} 100 \cdot \frac{\sin(\frac{5\pi k}{0,2} r)}{r} r^2 dr}{\int_0^{0,2} \frac{\sin(\frac{5\pi k}{0,2} r)}{r^2} \cdot r^2 dr} = \dots = 200 (-1)^{k+1} \cdot \frac{1}{5\pi k}$$

$$\Rightarrow u(r, t) = \sum_{k=1}^{\infty} 200 (-1)^{k+1} \cdot e^{-0,25\pi^2 k^2 t} \cdot \frac{\sin(5k\pi r)}{5k\pi r}$$

3.46

Spänningsskraften per l.e.

överallt är konstant = S .

a) Bestäm grundfrekvensen och beräkna

dess värde i fallet $a=1\text{ dm}$, $M=1,5\text{ g}$, $S=6\text{ N/m}$.

$$\left\{ \begin{array}{l} U_{tt}'' - c^2 \Delta U = U_{tt}'' - c(U_{xx}'' + U_{yy}'') = 0 \quad (\text{PDE}) \\ U(0, y, t) = U(a, y, t) = 0 \quad (\text{RV}) \\ U(x, 0, t) = U(x, 2a, t) = 0 \quad (\text{RV}) \end{array} \right.$$

Vi ser direkt att vi har operatorn:

$$A = -\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), D_A = \{U \in C^2(\Omega) | U = 0 \text{ på } \partial\Omega\}$$

PDE kan alltså skrivas $U_{tt}'' + c^2 Au = 0$ och vi vill ta fram egenvärden och egenvektorer till A .Variabelseparation: Definition:

$$\varphi(x, y) = X(x) \cdot Y(y)$$

$$-\Delta \cdot \varphi_k = \lambda_k \varphi_k$$

~~PDE~~ $-X'' \cdot Y - X Y'' = \lambda_k X Y$

$$\Leftrightarrow -\frac{X''}{X} - \frac{Y''}{Y} = \lambda_k \Rightarrow$$

$$\begin{cases} -\frac{X''}{X} = \mu \\ -\frac{Y''}{Y} = \nu \end{cases}$$

$$\lambda_k = \mu + \nu$$

$$\begin{cases} X'' + \mu X = 0 \\ X(0) = X(a) = 0 \end{cases}$$

$$\begin{cases} Y'' + \nu Y = 0 \\ Y(0) = Y(2a) = 0 \end{cases}$$

Vi har två
stycken diff-
ekvationer.

strunta i
triviala
lösningar.

Vi löser diffekvationerna...

$$X_i(x) = \sin\left(\frac{i\pi}{a}x\right), \quad \mu_i = \frac{i^2\pi^2}{a^2}$$

$$Y_k(y) = \sin\left(\frac{k\pi}{2a}y\right), \quad \nu_k = \frac{k^2\pi^2}{4a^2}$$

svaren.

$$\Rightarrow \lambda_{ik} = \left(\frac{i\pi}{a}\right)^2 + \left(\frac{k\pi}{2a}\right)^2 \text{ och } \varphi_{ik}(x,y) = \sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{k\pi y}{2a}\right)$$

Ansats: $U(x,y,t) = \sum_{k,i=1,1}^{\infty} U_{jk}(t) \varphi_{ik}(x,y)$

PDE: $U''_{jk} + c^2 \lambda_{ik} U_k = 0$

Egenvinkelfrekvensen är

$$\omega_{ik} = c \sqrt{\lambda_{ik}} = c \frac{\pi}{a} \sqrt{i^2 + \frac{k^2}{4}}$$

Grundfrekvensen är $f_{11} = \frac{\omega_{11}}{2\pi} = \frac{c \sqrt{\lambda_{11}}}{2\pi} = 50 \text{ Hz}$

b) Vilken är den lägsta egenfrekvens för vilken man har degeneration?

Vi vet att $f_{mn} = \frac{c}{2\pi} \sqrt{\lambda_{mn}}$ och vill att:

$$f_{m_1, n_1} = f_{m_2, n_2} \quad (\text{där } m_1 \neq m_2, n_1 \neq n_2)$$

$$\Leftrightarrow \sqrt{m_1^2 + \frac{n_1^2}{4}} = \sqrt{m_2^2 + \frac{n_2^2}{4}}$$

$$\Leftrightarrow 4m_1^2 + n_1^2 = 4m_2^2 + n_2^2$$

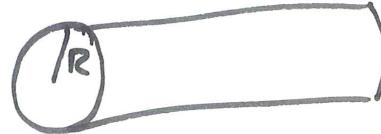
$$\text{Om } m_1 = 2 \text{ & } n_1 = 2 \Rightarrow 5 = 5 \Leftrightarrow m_2 = 1, \& n_2 = 4$$

(Vi kan visa att detta är minsta)

Svar: $f_{22} = f_{14}$

3.49

$$\begin{cases} u_t' - \alpha \Delta u = 0 \\ u(r, 0) = T_0 \\ u(R, t) = 0 \end{cases}$$



$$A = -\Delta = -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right)$$

$$A\psi_k = \lambda_k \psi_k \quad \text{Vi söker } \psi_k \text{ & } \lambda_k!$$

Insättning:

Dessels diff.ekv.

$$-\frac{1}{r} (r \cdot \psi_k')' = \lambda_k \psi_k \Leftrightarrow \psi_k'' + \frac{1}{r} \psi_k' + \lambda_k \psi_k = 0$$

$$\Rightarrow \psi_k = a J_0(\sqrt{\lambda_k} r) + b Y_0(\sqrt{\lambda_k} r)$$

↙ ψ_k är begränsad!

$$\psi_k(R) = 0 \Rightarrow \sqrt{\lambda_k} R = \alpha_{ok}$$

$$\Rightarrow \lambda_k = \left(\frac{\alpha_{ok}}{R} \right)^2, \quad \psi_k = a J_0 \left(\frac{\alpha_{ok}}{R^2} r \right)$$

Vi ansätter lösningen:

$$u(r, t) = \sum_{k=1}^{\infty} u_k \cdot \psi_k$$

$$\text{PDE: } \sum_{k=1}^{\infty} (u_k' - \alpha \lambda_k u_k) \psi_k = 0 \Rightarrow u_k = C_k \cdot e^{-\alpha \lambda_k t}$$

Detta ger:

$$u(r,t) = \sum_{k=1}^{\infty} c_k \cdot e^{-\alpha \left(\frac{K_{0k}}{R}\right)^2 t} \cdot J_0 \left(\left(\frac{K_{0k}}{R}\right)^2 r\right)$$

$$u(r,0) = T_0 \Rightarrow \sum_{k=1}^{\infty} c_k \varphi_k = T_0$$

Vi beräknar c_k genom att projicera T_0 på den ortogonala basen φ_k .

$$c_k = \frac{(\varphi_k | T_0)}{(\varphi_k | \varphi_k)} = \frac{\int_0^R J_0 \left(\left(\frac{K_{0k}}{R}\right)^2 r\right) T_0 \cdot r dr}{\int_0^R J_0 \left(\left(\frac{K_{0k}}{R}\right)^2 r\right) r dr}$$

Vi såg i början
att r är
viktfunktionen w .

3.51

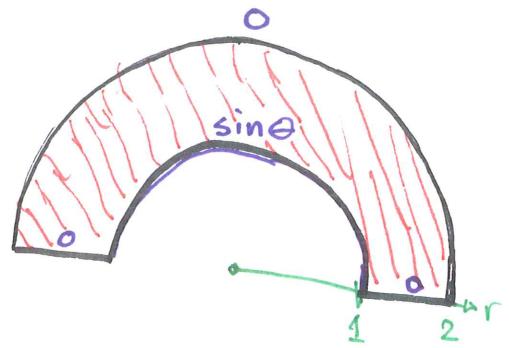
Lös randvärdesproblem.

PDE: $\Delta u = 0 , 1 < r < 2 , 0 < \theta < \pi$

$$u(1, \theta) = \sin \theta , u(2, \theta) = 0$$

RV:

$$u(r, 0) = 0 , u(r, \pi) = 0$$



$$\text{ex 3.19} \Rightarrow u(r, \theta) = \sum_{k=1}^{\infty} (a_k r^k + b_k r^{-k}) \sin(k\theta)$$

$$u(1, \theta) = \sum_{k=1}^{\infty} (a_k + b_k) \sin(k\theta) = \sin \theta \Rightarrow \begin{cases} a_1 + b_1 = 1 \\ a_k + b_k = 0 , k > 1 \end{cases}$$

$$u(2, \theta) = \sum (a_k \cdot 2^k + b_k \cdot 2^{-k}) \sin(k\theta) = 0 \Rightarrow 2^{2k} a_k + b_k = 0$$

$$K=1: \begin{cases} a_1 + b_1 = 1 \\ 2^2 a_1 + b_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -\frac{1}{3} \\ b_1 = \frac{4}{3} \end{cases}$$

$$K > 1: \begin{cases} a_k + b_k = 0 \\ 2^{2k} a_k + b_k = 0 \end{cases} \Leftrightarrow \begin{cases} a_k + b_k = 0 \\ 2^{2k} a_k - a_k = 0 \end{cases} \Leftrightarrow \begin{cases} a_k = 0 \\ b_k = 0 \end{cases}$$

SVAR

$$\Rightarrow u(r, \theta) = \left(-\frac{1}{3} r + \frac{4}{3} r^{-1} \right) \sin \theta = \frac{4y}{3(x^2+y^2)} - \frac{y}{3}$$

3.55

$$\begin{cases} \Delta u = 0, & 0 < r < 1, \quad 0 < \theta < 2\pi \\ u(r, 0) = 0, \quad u(r, \alpha) = 1 \\ u(1, \theta) = 1 \end{cases}$$

$$\tilde{u} = \frac{1}{\alpha} \theta, \quad v = u - \tilde{u}$$

Nya ekvationer:

$$\begin{cases} \Delta v = 0 \\ v(r, 0) = 0, \quad v(r, \alpha) = 1 - \frac{\theta}{\alpha} = 0 \\ v(1, \theta) = 1 - \frac{\theta}{\alpha} \end{cases}$$

Variabelseparation: $v(r, \theta) = R(r) \cdot \Theta(\theta)$

$$\Rightarrow \frac{1}{r} (r \cdot R') \Theta + \frac{1}{r^2} \Theta'' R = 0$$

$$\Leftrightarrow \frac{1}{r} (R' + r R'') \Theta + \frac{1}{r^2} \Theta'' R = 0$$

$$\Leftrightarrow -\frac{\Theta''}{\Theta} = \frac{r^2}{R} (R'' + \frac{1}{r} R') = \lambda$$

Vi har alltså:

$$\begin{cases} \Theta'' + \lambda \Theta = 0 & (1) \\ \frac{r^2}{R} (R'' + \frac{1}{r} R') = \lambda & (2) \end{cases} \quad \Theta(0) = 0, \quad \Theta(\alpha) = 0$$

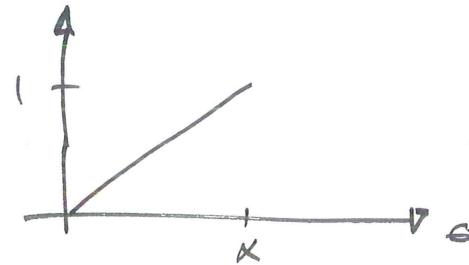
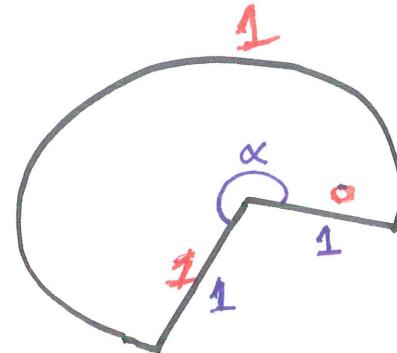
$$R(1) = 1 - \frac{\theta}{\alpha}$$

$$(1) \Rightarrow \Theta = a \sin(\sqrt{\lambda} \theta) + b \cos(\sqrt{\lambda} \theta)$$

$$\Theta(0) = 0 \Rightarrow b = 0$$

$$\Theta(\alpha) = 0 \Rightarrow \sqrt{\lambda} \alpha = k\pi \Rightarrow \lambda = \left(\frac{k\pi}{\alpha}\right)^2, \quad k = 0, 1, 2, \dots$$

$$\Rightarrow \Theta(\theta) = \sin\left(\frac{k\pi}{\alpha} \theta\right)$$



$$(2) \Rightarrow \frac{r^2}{R} (R'' + \frac{1}{r} R') = \lambda$$

$$\Leftrightarrow R'' + \frac{1}{r} R' - \frac{\lambda}{r^2} R = 0$$

Bessels diff.ekvation.

FS: $R(r) = ar^{k\pi/\alpha} + br^{-k\pi/\alpha}$ ($\lambda^* = (\frac{k\pi}{\alpha})^2$)

R är begr. $\Rightarrow b = 0$.

$$R(1) = a = 1 - \frac{\Theta}{\alpha}$$

$$\Rightarrow R(r) = \left(1 - \frac{\Theta}{\alpha}\right) \cdot r^{k\pi/\alpha}$$

$$\Rightarrow u(r, \theta) = v + \tilde{u} = R(r)\Theta(\theta) + \frac{\Theta}{\alpha} = \frac{\Theta}{\alpha} + \sum_{n=1}^{\infty} \left(1 - \frac{\Theta}{\alpha}\right) r^{k\pi/\alpha} \cdot \sin\left(\frac{k\pi}{\alpha} \theta\right)$$

3.62 Lös Dirchlet-problemet i sfäriska koordinater.

$$\Delta u = 0$$

$$u(1, \theta, \phi) = \cos^2 \theta$$

$$s = \cos \theta$$

u är beroende av ϕ , ansats: $u = u(r, \theta) = R(r) S(s)$

$$\begin{cases} \Delta u = \frac{1}{r}(ru)'' + \frac{1}{r^2}(((1-s^2)u_s')_s)' = 0 \\ u(1, s) = s^2 \end{cases}$$

$$\text{PDE} \Rightarrow \Delta u = \frac{1}{r^2} (r^2 u_r')' + \frac{1}{r^2} ((1-s^2)u_s')_s' = 0$$

$$\Leftrightarrow \frac{(r^2 R')'}{R} = - \frac{((1-s^2)s')'}{s} = \lambda = \text{konstant.}$$

Detta ger oss två diffekvationer:

$$\begin{cases} (r^2 R')' - \lambda R = 0 & , 0 < r < 1 \\ (1-s^2)S'' - 2sS' + \lambda S = 0 & -1 < s < 1 \end{cases}$$

Lös dessa
och du
HAR
SVARET!

$S(s)$: Enligt sats 5.4 har vi endast begr. lösning för:

$$\lambda_l = l(l+1), l \in \mathbb{N}$$

$$l = P_l(s)$$

$R(r)$:

$$R'' + \frac{2}{r} R' - \frac{l(l+1)}{r^2} R = 0$$

FS: $\boxed{R(r) = ar^l + br^{l-1}}$, $l = 0, 1, 2, \dots$

\uparrow begränsad

Vi har alltså

$$\boxed{U(r, s) = \sum_{k=0}^{\infty} a_k \cdot r^k \cdot P_k(s)}$$

$$\begin{aligned} P_0 &= 1 \\ P_1 &= s \\ P_2 &= \frac{1}{2}(3s^2 - 1) \end{aligned}$$

$$U(1, s) = \sum_{k=0}^{\infty} a_k P_k(s) = s^2 \Rightarrow$$

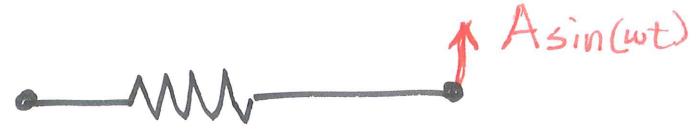
$$\begin{aligned} a_0 &= \frac{1}{2} \\ a_1 &= 0 \\ a_2 &= \frac{2}{3} \\ a_k &= 0, k > 2 \end{aligned}$$

SVAR: $\boxed{U(r, s) = \frac{1}{3} + \frac{2}{3} r^2 P_2(s)}$

3.66

En stav eller fjäder.

$$a) \left\{ \begin{array}{l} U_{tt}'' - c^2 U_{xx}'' = 0 \\ U(0,t) = 0 \end{array} \right.$$



$$\left. \begin{array}{l} KU_x'(L,t) = Asin(wt) \end{array} \right.$$

b) Avgör vilka frekvenser det finns en periodisk lösning.

Vi ansätter: $U(x,t) = H(x) \cdot A \sin(wt)$

$$\text{PDE: } (H''c^2 - H\omega^2) A \sin(wt) = 0$$

$$\Rightarrow H'' - \frac{H\omega^2}{c^2} = 0 \Leftrightarrow H = a \sin\left(\frac{\omega}{c}x\right) + b \cos\left(\frac{\omega}{c}x\right)$$

$$H(0) = 0 \Rightarrow b = 0 \Rightarrow H = a \sin\left(\frac{\omega}{c}x\right)$$

$$KU_x'(L,t) = Asin(wt) \Rightarrow K H_x'(L) \cancel{A} \sin\left(\frac{\omega}{c}x\right) = Asin(wt)$$

$$\Leftrightarrow K \frac{\omega}{c} a \cos\left(\frac{\omega}{c}L\right) A \sin\left(\frac{\omega}{c}x\right) = A \sin(wt)$$

$$\Rightarrow \frac{ak\omega}{c} \cos\left(\frac{\omega}{c}L\right) = 1 \Rightarrow a = \frac{c}{kw \cos\left(\frac{\omega}{c}L\right)}$$

Nu har vi $U(x,t)$:

$$U(x,t) = \frac{A c \cdot \sin\left(\frac{\omega}{c}x\right)}{kw \cos\left(\frac{\omega}{c}L\right)} \sin(wt)$$

$$\cos\left(\frac{\omega}{c}L\right) \neq 0 \Leftrightarrow w \neq \frac{c}{L} \left(\frac{\pi}{2} + \pi k \right)$$

Resonans!

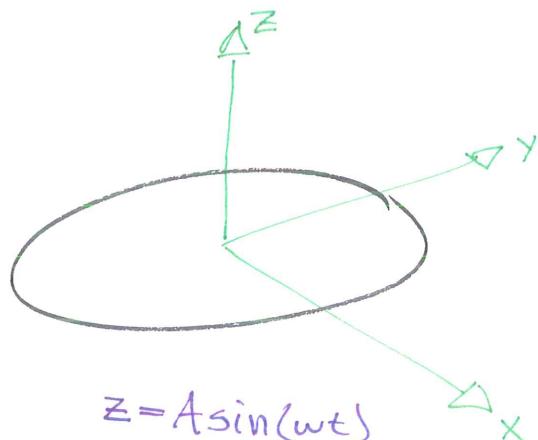
3.69

Bestäm membranets stationära periodiska svängning.

$$\text{PDE} \quad U_{rr}'' - c^2 \Delta u = 0$$

$$\text{RV} \quad U(R, t) = A \sin(\omega t) = A \cdot \text{Im}(e^{i\omega t})$$

$$\text{BN} \quad U(r, 0) = 0$$



$$\text{Ansats: } U(r, t) = H(r) \cdot e^{i\omega t}$$

$$\text{PDE: } (i\omega)^2 H(r) e^{i\omega t} - c^2 \frac{1}{r} (r H')' e^{i\omega t} = 0$$

$$\Leftrightarrow \left(-\omega^2 H - \frac{c^2}{r} (r H')' \right) e^{i\omega t} = 0$$

$$\Leftrightarrow -\omega^2 H - \frac{c^2}{r} H' - \frac{c^2}{r} r H'' = 0$$

$$\Leftrightarrow H'' + \frac{1}{r} H' + \frac{\omega^2}{c^2} H = 0$$

$$\Rightarrow H = a J_0 \left(\sqrt{\frac{\omega^2}{c^2}} r \right) = a J_0 \left(\frac{\omega}{c} r \right)$$

$$U(R, t) = A \text{Im}(e^{i\omega t}) = H \cdot e^{i\omega t}$$

$$\Rightarrow A = a J_0 \left(\frac{\omega}{c} R \right) \Leftrightarrow a = \frac{A}{J_0 \left(\frac{\omega}{c} R \right)}$$

$$\Rightarrow U(x, t) = A \frac{J_0 \left(\frac{\omega r}{c} \right)}{J_0 \left(\frac{\omega R}{c} \right)} \sin(\omega t)$$

Vi får resonans då $J_0 \left(\frac{\omega R}{c} \right) = 0$

$$\Rightarrow \alpha_{0k} \neq \frac{\omega R}{c}$$

$$\Leftrightarrow \omega \neq \frac{c \alpha_{0k}}{R}$$