Integration av vektorvärda funktioner

$\vec{A}(u,v,\dots)$ kontinuerlig i D

Satser

* $\iint_D \vec{A}\, du\, dv \dots = \left(\iint_D A_x\, du\, du \dots + \iint_D A_y\, du\, du \dots \right)$
* $\int_a^b \frac{d\vec{A}}{du}\, du = \vec{A}(b) - \vec{A}(a)$
* $\frac{d}{dx} \int_a^x \vec{A}(u)\, du = \vec{A}(x)$

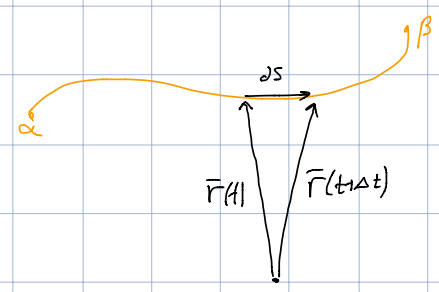
Linje- och ytintegraler

* Längden av en kurva

$\vec{x} = \vec{x}(t) = (x_1(t), \dots, x_n(t))$

$n=2$: $\vec{r}_{2D} = (x(t), y(t))$

Längden av en kurva: $\vec{r}(t)$ $\alpha \le t \le \beta$



$\Delta S = |\vec{r}(t+\Delta t) - \vec{r}(t)| = |\vec{r}\,'(t)|\, dt$

$S(t) = \int_\alpha^t ds = \int_\alpha^t |\vec{r}\,'(\tau)|\, d\tau$

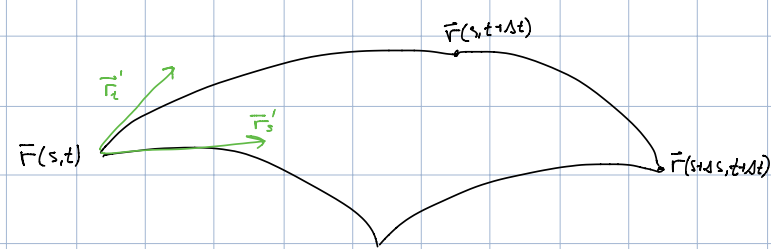
* Arean av en yta.

$\vec{r}(s,t)$, $(s,t) \in D$

Vad är dA?

helt enkelt: $dA = |\vec{r}_s\,' \times \vec{r}_t\,'|\, ds\, dt$

$A_Y = \iint_Y dA$



* Kurvintegraler

$\vec{F}(\vec{r})$ påverkar en partikel

$\vec{r}(t)$ $\alpha \le t \le \beta$

$\vec{F} \cdot d\vec{r} = \vec{F} \cdot \vec{r}\,'(t)\, dt$

$\int_\alpha^\beta \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ ← Kurvintegralen

Definition (i 2D)

Låt $\vec{F}(\vec{r}) = (P(\vec{r}), Q(\vec{r})) = (P(x,y), Q(x,y))$ vara kont. i D

Om γ är orienterad C^1 kurva, $\vec{r}(t) = (x(t), y(t))$, så kallar vi

$$\int_\alpha^\beta \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_\alpha^\beta \big(P(x,y)\,x'(t) + Q(x,y)\,y'(t)\big)\,dt$$

Kurvintegralen av $\vec{F}$ längs γ

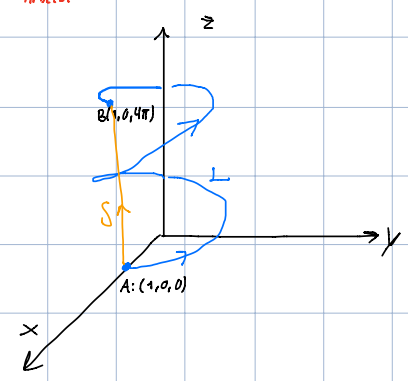
$\int \vec{F}\,d\vec{r} = \int P\,dx + Q\,dy$ kallas differentialform

$$\vec{r}\,'(t)\,dt = \underbrace{\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}_{\hat{T}\text{: kurvans enhetstangent}} |\vec{r}\,'(t)|\,dt \longrightarrow I_\gamma = \int_\gamma \underbrace{\vec{F}\cdot\hat{T}\,ds}_{\text{Arbetet}}$$

Exempel

$$\begin{cases} L \text{ parametriseras } \vec{r}(\cos u, \sin u, u) \\ \vec{F} = (-yz, xz, 2) \end{cases}$$

Vad blir arbetet från A till B?

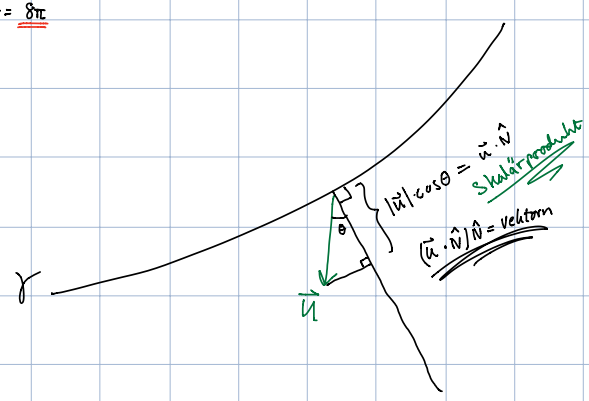


Längs L: $\displaystyle\int_0^{4\pi} \vec{F}(\vec{r}(u)) \frac{d\vec{r}}{du}\,du = \Big[\vec{F}(\vec{r}(u)) = (-u\sin u, u\cos u, 2)\Big] = \int_0^{4\pi} (-u\sin u, u\cos u, 2)(-\sin u, \cos u, 1)\,du = \int_0^{4\pi} u + 2\,du = 8\pi(\pi+1)$

Längs S: $\displaystyle\int_0^{4\pi} \vec{F}\,d\vec{r} = \Big[\vec{r} = (1,0,v),\ 0 \le v \le 4\pi\Big] = \int_0^{4\pi} 2\,dv = 8\pi$

Strömningslära

$$\int_\gamma \underbrace{\vec{u}}_{\text{vektorfält}} \cdot \hat{N}\,ds$$



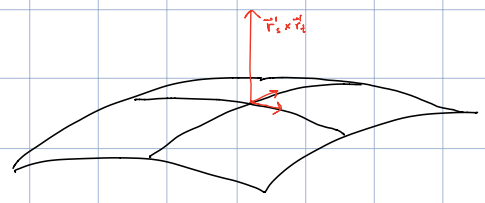
$\vec{u}\cdot\hat{N}\cdot ds$ = materieströmning genom ds per tidsenhet

$I = \int_\gamma \vec{u} \cdot \hat{N}\, ds$ beskriver flödet genom γ

$$\int_\gamma \vec{u} \cdot \hat{N}\, ds = \left[\hat{T} = (dx, dy) \Rightarrow \hat{N} = (dy, -dx),\ \vec{u} = (u_1, u_2) \right] = \underline{\underline{\int_\gamma (u_1 dy - u_2 dx)}}$$

differentialform.

Ytintegraler



Parameterframställningen bestämmer orienteringen.

Strömtäthetsvektor $\vec{u}$

Flödet genom den lilla arean dS: $\vec{u} \cdot \hat{N}\, ds$

Flödet genom hela ytan: $\boxed{\Phi = \iint_\gamma \vec{u} \cdot \underbrace{\hat{N} \cdot dS}_{dS}}$