

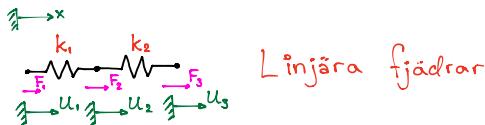
Föreläsning 2

"Idag ska vi snaka hela föreläsningen om hur man kan sätta ihop två fjädrar"

Direkt metod

- Dela upp problemet
- Ställ upp konstitutiva samband
- Sammanställ

Tvåfjädersystem



Elementbeskrivning

$$\begin{matrix} N \\ \downarrow \\ u_i^e \end{matrix} \quad \begin{matrix} M \\ \downarrow \\ u_2^e \end{matrix}$$

Anmärkning: $N = N(\delta) = k\delta$ (Vi gör livet enkelt)

Konstitutiv parameter

Vi ritar om lite:

$$\begin{matrix} P_i^e \\ \downarrow \\ u_i^e \end{matrix} \quad \begin{matrix} M \\ \downarrow \\ u_2^e \end{matrix}$$

$$\text{Vi får ekvationer: } \begin{cases} N = k\delta = N(u_i^e - u_2^e) \\ P_i^e = -N = -k(u_2^e - u_i^e) \\ P_2^e = N = k(u_2^e - u_i^e) \end{cases}$$

VIKTIGT

positivt definit $\Rightarrow \det(M) > 0 \Rightarrow M$ inverterbar
pos. semidefinit $\Rightarrow \det(M) = 0 \Rightarrow M$ ej inverterbar

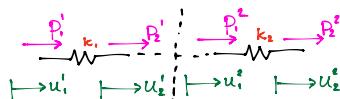
Direct stiffness method



As one of the methods of **structural analysis**, the **direct stiffness method**, also known as the **matrix stiffness method**, is particularly suited for computer-automated analysis of complex structures including the **statically indeterminate** type. It is a **matrix** method that makes use of the members' stiffness relations for computing member forces and displacements in structures. The direct stiffness method is the most common implementation of the **finite element method** (FEM). In applying the method, the system must be modeled as a set of simpler, idealized elements interconnected at the nodes. The material stiffness properties of these elements are then, through **matrix mathematics**, compiled into a single matrix equation which governs the behaviour of the entire idealized structure. The structure's unknown displacements and forces can then be determined by solving this equation. The direct stiffness method forms the basis for most commercial and free source finite element software.

The direct stiffness method originated in the field of **aerospace**. Researchers looked at various approaches for analysis of complex airplane frames. These included **elasticity theory**, **energy principles in structural mechanics**, **flexibility method** and **matrix stiffness method**. It was through analysis of these methods that the direct stiffness method emerged as an efficient method ideally suited for computer implementation.

Nu lägger vi ihop fjädrarna:



$$\text{Matrisform: } \underbrace{\begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_K \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_a = \underbrace{\begin{bmatrix} P_1^e \\ P_2^e \\ 0 \end{bmatrix}}_f$$

Anmärkning: Symmetriska
 $\det(K) = 0$
 \Rightarrow ej inv. bar

Kinematisk kompatibilitet

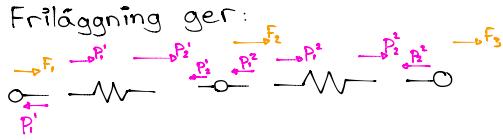
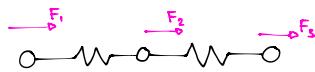
$$u'_1 = u_1, \quad u'_2 = u_2 = u^2, \quad u'_3 = u_3$$

Vi expanderar matrisekvationerna ovan.

$$\underbrace{\begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{K^{ee}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_a = \underbrace{\begin{bmatrix} P_1^e \\ P_2^e \\ 0 \end{bmatrix}}_f$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix}}_{K_2^{ee}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_a = \underbrace{\begin{bmatrix} 0 \\ P_1^e \\ P_2^e \end{bmatrix}}_f$$

Jämvikt



Kraftjämvikt vid noderna:

$$\textcircled{1} \quad F_1 - P_1' = 0$$

$$\textcircled{2} \quad F_2 - P_2' - P_1^2 = 0 \quad , \text{ Matriseku: } \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} P_1' \\ P_2' + P_1^2 \\ P_2^2 \end{bmatrix} = \begin{bmatrix} P_1' \\ P_2' \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ P_2^2 \\ P_2^2 \end{bmatrix} \Rightarrow \mathbb{F} = f_1^{ee} + f_2^{ee}$$

$$\textcircled{3} \quad F_3 - P_2^2 = 0$$

Kombinera med K_1 och K_2 !

$$\left. \begin{array}{l} K_1^{ee} \alpha = f_1^{ee} \\ K_2^{ee} \alpha = f_2^{ee} \end{array} \right\} \Rightarrow K_1^{ee} \alpha + K_2^{ee} \alpha = f_1^{ee} + f_2^{ee} = \mathbb{F} = \underbrace{(K_1^{ee} + K_2^{ee})}_{K} \alpha = \boxed{K \alpha}, \quad K = \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 + K_2 & -K_2 \\ 0 & -K_2 & K \end{bmatrix}$$

$$\det(K) = 0 \Rightarrow \text{ej inverterbar.}$$

Kvadratisk form I

$$I = \mathbb{F}^T K \mathbb{F} = [U_1, U_2, U_3] \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 + K_2 & -K_2 \\ 0 & -K_2 & K \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \dots = K_1(U_1 - U_2)^2 + K_2(U_2 - U_3)^2$$

Kvadratisk form I

$$I = \alpha^T K \alpha = [U_1, U_2, U_3] \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 + K_2 & -K_2 \\ 0 & -K_2 & K \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \dots = K_1(U_1 - U_2)^2 + K_2(U_2 - U_3)^2$$

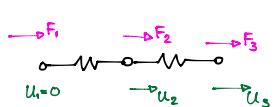
Slutsats: $\boxed{I \geq 0 \quad \forall \alpha}$

$I = 0$ t.ex. om $\alpha = [1 \ 1 \ 1]^T$

$\Rightarrow K$ är positivt semidefinit $\Leftrightarrow \det(K) = 0$

Randvillkor

-exempel: $U_1 = 0$



Du kan inte veta både Neumann- och Dirichletvillkor

$$\begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 + K_2 & -K_2 \\ 0 & -K_2 & K \end{bmatrix} \begin{bmatrix} 0 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}}_K \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}$$

$$\mathbb{X}^T \tilde{K} \mathbb{X} = K_1 X_1^2 + K_2 (X_2 - X_3)^2 > 0 \Rightarrow \det(\tilde{K}) \neq 0 \Leftrightarrow \tilde{K} \text{ är inverterbar!}$$

⚠ Genom att lägga på ett randvillkor får vi ett lösbart system!

Kan vi använda samma metod för svårare problem?

Målet är att sätta upp en styrhetsmatris.

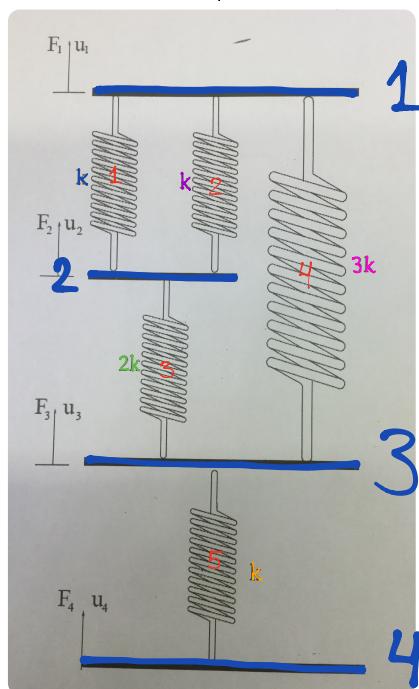
$$\text{Elementstyrhetsmatris: } K_i^e = \begin{bmatrix} K_i & -K_i \\ -K_i & K_i \end{bmatrix}$$

Matris som beskriver hur elementen

hänger ihop: *från* *til*

$$Edof = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ 4 & 1 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

Vilket element *vår ska de sitta?*



Vi lägger in enhetsstyrhetsmatriser enligt indexeringen ovan

$$\begin{bmatrix} K+K+3K & -K-K & -3K & 0 \\ -K-K & K+K+2K & -2K & 0 \\ -3K & -2K & 2K+3K+K & -K \\ 0 & 0 & -K & K \end{bmatrix} = \begin{bmatrix} 5K & -2K & -3K & 0 \\ -2K & 4K & -2K & 0 \\ -3K & -2K & 6K & -K \\ 0 & 0 & -K & K \end{bmatrix}$$

Med denna metod ser vi att det inte är så svårt att beskriva mer komplicerade system!