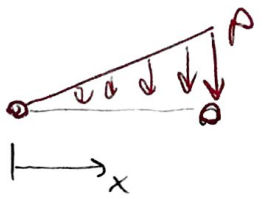
Föreläsning FEM

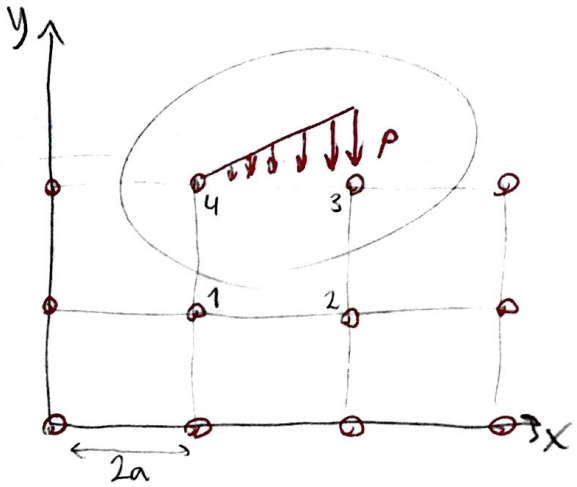
Exempel FE-formulering av elastiska problem forts

$$\int_V (\tilde{\nabla}\mathbb{v})^T \boldsymbol{\sigma}\, dV = \int_S \mathbb{v}^T \mathbb{t}\, dS + \int_V \mathbb{v}^T \mathbb{b}\, dV \qquad \text{svag form}$$

$$\underbrace{\mathbb{u} = \mathbb{N}\mathbb{a}, \quad \mathbb{v} = \mathbb{N}\mathbb{c}, \quad \boldsymbol{\varepsilon} = \tilde{\nabla}\mathbb{u} = \mathbb{B}\mathbb{a}, \quad \tilde{\nabla}\mathbb{v} = \mathbb{B}\mathbb{c}, \quad \boldsymbol{\sigma} = \mathbb{D}\cdot\boldsymbol{\varepsilon}}_{\text{insättning av dessa ger som behandt.}}$$

$$\int_V \mathbb{B}^T \mathbb{D} \mathbb{B}\, dV\, \mathbb{a} = \int_S \mathbb{N}^T \mathbb{t}\, dS + \int_V \mathbb{N}^T \mathbb{b}\, dV$$





Vi får $\mathbb{t}$-matrisen till

$$\mathbb{t} = [t_x \;\; t_y]^T = \left[0 \;\; -\frac{x}{2a}p\right]^T$$

som randspänning!

$$\mathbb{f}_b = t\int_L \underbrace{\begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e \end{bmatrix}^T}_{2\times 4 \text{ ty } 4\text{-nods-element}} \cdot \begin{bmatrix} 0 \\ t_y \end{bmatrix} dL =$$

$$= t\left[0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \int_0^{2a} N_3^e t_y\, dx \;\; 0 \;\; \int_0^{2a} N_4^e t_y\, dx\right]$$

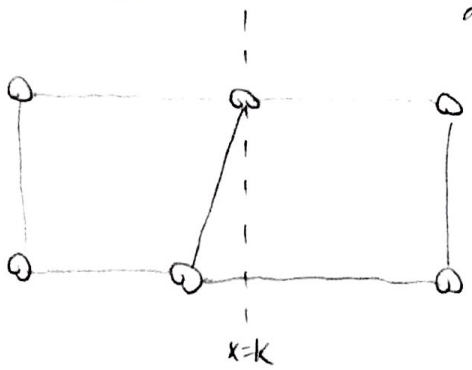
(N_3^e och N_4^e kan lätt inses utan någon C-metod längs randen)

$$= t\left[0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; -\tfrac{2}{3}pat \;\; 0 \;\; -\tfrac{1}{3}pat\right] \quad \text{med} \quad \begin{cases} N_3^e = \frac{x}{2a} \\ N_4^e = 1 - \frac{x}{2a} \end{cases}$$

Är detta rimligt? --- Ja! $\boxed{\text{klara!}}$

Kapitel 19

Lite repetition...



$T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$

[insert $x=k$]

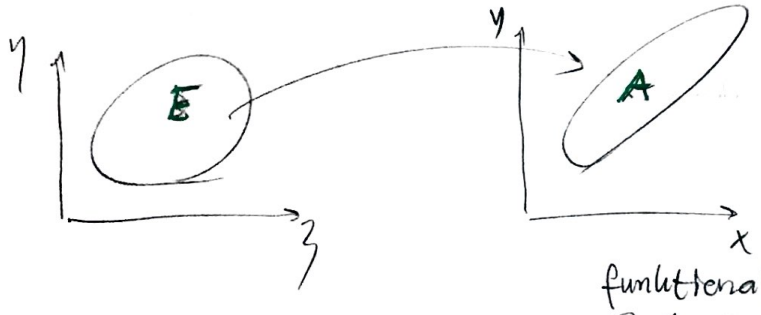
$\Downarrow$

$T = \alpha_1' + \alpha_2' x$ ← kompatibel

Men, [insert $y = kx + m$] $\Rightarrow$ $T = \alpha_1' + \alpha_2' x + \alpha_3' x^2$ ← Ej unik lösning!

Hur löser vi detta problem om vi vill kunna använda fyrnodiga element som inte är kvadratiska?

Flerdim ger svaret:



$$\begin{cases} x = x(\xi, \eta) \\ y = y(\xi, \eta) \end{cases}$$

$$\int_A f(x,y)\,dx\,dy = \int_E f(x(\xi,\eta), y(\xi,\eta)) \cdot \underbrace{|J(\xi,\eta)|}_{\text{funktionaldeterminant.}}\,d\xi\,d\eta$$

Varför vill vi kunna göra denna mappning? Vi kommer att göra ett speciellt val på $x = x(\xi,\eta)$ och $y = y(\xi,\eta)$

$\int_A B^T D B\,dx\,dy = [8 \times 8]$-matris i detta fall.

$$\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases} \Rightarrow \begin{aligned} dx &= \frac{\partial x}{\partial \xi} d\xi + \frac{dx}{\partial \eta} d\eta \\ dy &= \frac{dy}{d\xi} d\xi + \frac{dy}{d\eta} d\eta \end{aligned} \Rightarrow \begin{bmatrix} dx \\ dy \end{bmatrix} = \overbrace{\begin{bmatrix} \frac{dx}{\partial \xi} & \frac{dx}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{dy}{d\eta} \end{bmatrix}}^{J\text{ - jacobian}} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}$$

$\det J > 0$

$$d\eta = 0 \implies \mathbb{a} = \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{bmatrix} d\xi$$

$$d\xi = 0 \implies \mathbb{b} = \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \end{bmatrix} d\eta$$

$$dA = |\mathbb{a} \times \mathbb{b}| = \left| \det \begin{bmatrix} \frac{\partial x}{\partial \xi} d\xi & \frac{\partial x}{\partial \eta} d\eta \\ \frac{\partial y}{\partial \xi} d\xi & \frac{\partial y}{\partial \eta} d\eta \end{bmatrix} \right|$$

$$= d\xi \cdot d\eta \, |\det(\mathbb{J})|$$

kräver att $\det \mathbb{J} > 0$! ← ?

- [Nu har Matthias skrivit typ tre tavlor med anteckningar på en minut...]

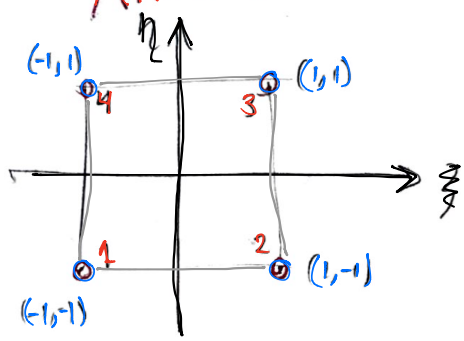
- ## 4-nods-element

Vi vet att elementränder parallella med x & y-axlarna uppfyller kompatibilitetskraven. Hmm...

Idé: Definiera elementen i ξ-η-domänen med elementränder parallella med ξ och η!

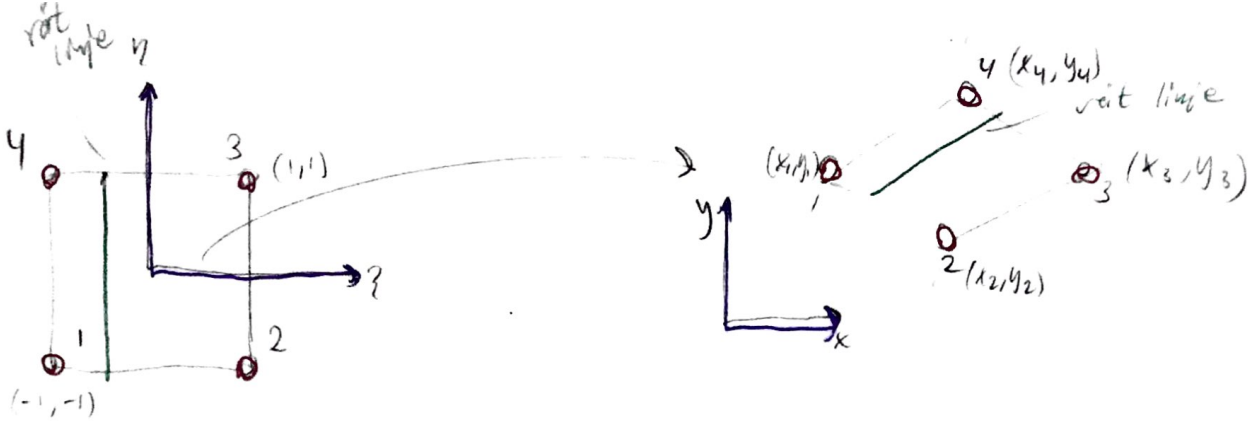
- $$N_1^e = \frac{1}{4}(\xi - 1)(\eta - 1)$$
$$N_2^e = \frac{1}{4}(\xi + 1)(\eta - 1)$$
- $$N_3^e = \frac{1}{4}(\xi + 1)(\eta + 1)$$
$$N_4^e = \frac{1}{4}(\xi - 1)(\eta + 1)$$

Ansatz



$$T = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \cdot \eta$$

Så, $T = \mathbb{N}^e \mathbb{a} = N_1^e a_1 + N_2^e a_2 + N_3^e a_3 + N_4^e a_4 = T(\xi, \eta)$



Avbildningen $x(\xi,\eta) = N_1^e x_1 + N_2^e x_2 + N_3^e x_3 + N_4^e x_4 = \underline{N^e x^e}$

$y(\xi,\eta) = N_1^e y_1 + N_2^e y_2 + N_3^e y_3 + N_4^e y_4 = \underline{N^e y^e}$

Detta är en <u>isoparametrisk</u> avbildning

<u>Hörnpunkt:</u>

$$x(-1,-1) = \underbrace{N_1^e(-1,-1)}_{=1} x_1 + \underbrace{N_2^e(-1,-1)}_{0} x_2 + \underbrace{N_3^e(-1,-1)}_{0} x_3 + \underbrace{N_4^e(-1,-1)}_{0} x_4 = x_1$$

⚠ Slutsats av detta lika exempel: En <u>hörnpunkt</u> avbildas på en <u>hörnpunkt</u>

<u>Rät linje:</u>

Notera att $\begin{cases} x(\xi,\eta) = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta \\ y(\xi,y) = \beta_1 + \beta_2\xi + \beta_3\eta + \beta_4\xi\eta \end{cases}$

Rät linje i ξ-η-systemet fås enkelt av ex: $\xi = k$

$\xi = k \Rightarrow \quad x = \alpha_1' + \alpha_2'\eta$

$\qquad\qquad\quad y = \beta_1' + \beta_2'\eta$

⚠ Slutsats: <u>Räta linjer</u> avbildas på <u>räta linjer</u>

Okej...

Är completeness uppfyllt då?

$$T = N^e a = \sum N_i^e a_i$$

$$\begin{cases} x = N^e x^e \\ y = N^e y^e \end{cases}$$

Vi vill kunna representera konstant värde och konstant gradient,

dvs: $T = \alpha_1 + \alpha_2 x + \alpha_3 y$,

$$T = \alpha_1 + \alpha_2 \underline{\underline{N^e x^e}} + \alpha_3 \underline{\underline{N^e y^e}}$$

- Välj specvellt att nodtemperaturen ges av

$$a_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i$$

- Insättning:

$$T = \sum N_i a_i = \sum N_i a_1 + \sum N_i a_2 x_i + \sum N_i a_3 y_i = \underline{\alpha_1 + \alpha_2 x + \alpha_3 y}$$

→ Jappo!

→ Då har vi konvergens!

- $$\mathbb{K} = \int_A t\, \mathbb{B}^T \mathbb{D} \mathbb{B}\, dA = \int_A t \underbrace{\begin{bmatrix} \frac{\partial N^T}{\partial x} & \frac{\partial N^T}{\partial y} \end{bmatrix}}_{?} \cdot \mathbb{D} \begin{bmatrix} \frac{dN}{dx} \\ \frac{dN}{dy} \end{bmatrix} dx\, dy$$

- Så
$$\left.\begin{aligned} \frac{\partial N}{\partial \xi} &= \frac{\partial N}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial N}{\partial \eta} &= \frac{\partial N}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N}{\partial y}\frac{\partial y}{\partial \eta} \end{aligned}\right\} \begin{bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\mathbb{J}^T \text{ - aha!}} \cdot \begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial N}{\partial x} & \frac{\partial N}{\partial y} \end{bmatrix}^T = (\mathbb{J}^T)^{-1} \cdot \begin{bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{bmatrix}$$

$$\Rightarrow \boxed{\mathbb{K}^e = \int_{-1}^{1}\int_{-1}^{1} t \begin{bmatrix} \frac{\partial N^T}{\partial \xi} & \frac{\partial N^T}{\partial \eta} \end{bmatrix} \cdot \mathbb{J}^{-1} \cdot \mathbb{D} (\mathbb{J}^T)^{-1} \begin{bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{bmatrix} \cdot |\det(\mathbb{J})|\, d\xi\, d\eta.}$$

Knepigt att integrera!

⇒ KAP 2.0 Numerisk integration