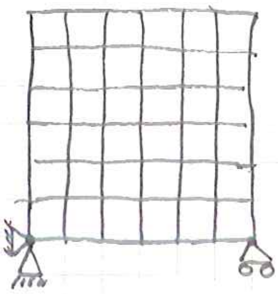
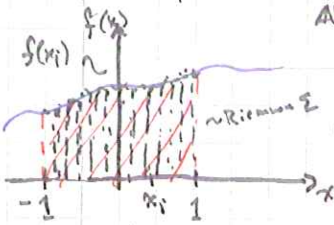
7/5-2013 **KAP20. (Num. integration)**

$$\mathbb{K}^e = \underbrace{\int_{-1}^{1}\int_{-1}^{1} \left[\frac{\partial N^e}{\partial \xi}\ \frac{\partial N^e}{\partial \eta}\right] \mathbb{J}^{-1} \underset{\approx}{\mathbb{D}} (\mathbb{J}^{-1})^T \begin{bmatrix}\frac{\partial N^e}{\partial \xi} \\ \frac{\partial N^e}{\partial \eta}\end{bmatrix} \det(\mathbb{J})\, d\xi\, d\eta}_{\text{sjukt tung att integrera... :(}}$$



$$\left(\mathbb{K}^e = [\text{projektet}] = \int_{A^e} t\,\mathbb{B}^{eT}\mathbb{D}\,\mathbb{B}^e\, dA = \mathbb{B}^{eT}\mathbb{D}\,\mathbb{B}^e t A^e\right)$$

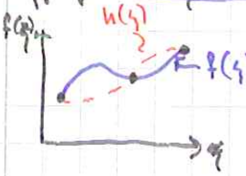


$$I = \int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} f(x_i) H_i$$

- **Newton-Cotes:**

Idé: Istället för att integrera $f(x)$ tittar vi en funktion $h(x)$ som "liknar" $f(x)$.

Välj h som ett polynom av grad $n-1$, som uppfyller $\boxed{h(\xi_i) = f(\xi_i)}$ $i = 1, 2, \dots n$



Ta fram $h(\xi)$

$(h = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2)$

Lagrangepolynom

$$h(\xi) = \underbrace{\frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} f(\xi_1)}_{\text{Term 1 } l_1^{n-1}} + \underbrace{\frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)}}_{\text{Term 2 } l_2^{n-1}} f(\xi_2) + \underbrace{\frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)}}_{\text{Term 3. } l_3^{n-1}} f(\xi_3) =$$

$= \sum_{k=1}^{n} f(\xi_k)\, l_k^{n-1}(\xi_k) \Rightarrow h(\xi)$ är ett polynom som passerar $(\xi_i, f(\xi_i))$.

$$I = \int_{-1}^{1} f(x)\,dx \approx \int_{-1}^{1} h(x)\,dx = \int_{-1}^{1} \sum_k f(\xi_k)\, l_k^{n-1}(\xi_k)\, d\xi =$$

$$= \sum f(\xi_k) \underbrace{\int_{-1}^{1} l_k^{n-1}(\xi)\, d\xi}_{H_k}$$

|| Notera: med 3 integrationspunkter kan vi integrera ett polynom av grad 2 korrekt.

H_k - Beräknas med penna & papper.

|| Notera Integrationspunkterna valdes fritt! Är detta optimalt?

- **Gauß-integration**

Antag att vi har ett polynom av grad $2n-1$, dvs $2n$ parametrar.

$$g(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \dots + \alpha_{2n}\xi^{2n-1}$$

$$I^A = \int_{-1}^{1} g(\xi)\,d\xi = 2\alpha_1 + \underset{\text{Alla udda termer försvinner.}}{0} + \frac{2}{3}\alpha_3 + \dots + \cancel{\frac{2}{2n-1}}\alpha_{2n}$$

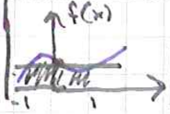
Allmän integrationsformel

$$I^B = \sum_{k=1}^{n} g(\xi_k) H_k \quad \boxed{I^A = I^B} \ ; \ I^B = \alpha_1 \sum_1^n H_k + \alpha_2 \sum_1^n \xi_k H_k + \alpha_3 \sum_1^n H_k \xi_k^2 \dots$$

$$2 = \alpha_1 \sum_1^n H_k$$
$$0 = \sum_1^n \xi_k H_k$$
$$\frac{2}{3} = \sum_1^n \xi_k^2 H_k$$

Antal ekvationer: $2n$

Antal obekanta: $n\ \xi_k$, $n\ H_k$ } $2n$



|| Notera: Gauß integration: Med n integrationspunkter kan vi integrera ett polynom av grad $2n-1$ exakt!

$$I = \int f(x)\,dx = \int_{-1}^{1} f(\xi)\frac{dx}{d\xi}\,d\xi = [\text{t.ex. 2 integ.punkt.}] =$$

$$= \left(f(\xi)\frac{dx}{d\xi}\right)\Big|_{-0{,}577} \cdot 1 + \left(f(\xi)\frac{dx}{d\xi}\right)\Big|_{+0{,}577} \cdot 1.$$

$$\mathbb{K}^e = \int_{-1}^{1}\int_{-1}^{1} \left[\frac{dN^e}{d\xi}\ \frac{dN^e}{d\eta}\right] \mathbb{J}\,\mathbb{D} \dots d\xi\, d\eta$$

* Full integration
* Reducerad integration

(4-nod element) Full integration $\begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}$ $n = 2$ (i varje led)

Reducerad integration $\boxed{\circ}$