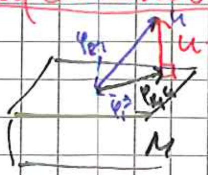
8/2-2012 (9) **Repe:** H prehilbertrum – linjärt rum med skalärprod

Projektion $\varphi_1, \dots, \varphi_n$ parvis ortogonala, $M = [\varphi_1, \dots, \varphi_n]$ $v \in M$, $v = \sum_{k=1}^{n} c_k \varphi_k$ $c_k \in \mathbb{R}$

$$u \in H \quad P_M u = \sum_{k=1}^{n} \frac{(\varphi_k|u)}{\|\varphi_k\|^2} \varphi_k$$

Projektionssatsen
1) $P_M u \in M$
2) $u - P_M u \in M^{\perp}$
3) $\inf_{v \in M} \|u - v\|^2 = \|u - P_M u\|^2 = \|u\|^2 - \sum_{k=1}^{n} \left| \frac{(\varphi_k|u)}{\|\varphi_k\|^2} \right|^2$



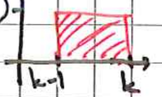
Ex) Betrakta skalärprodukt

$$(u|v) = \int_0^\infty u(x)\, v(x)\, e^{x} dx$$

viktfunktion

får norm $\|u\|^2 = \int_0^\infty |u(x)|^2 e^x dx$

Då blir $\varphi_k(x) = \Theta(x-(k-1)) - \Theta(x-k)$, $k = 1, 2, 3, \dots$



ortogonala $(\varphi_j|\varphi_k) = 0$

Bestäm a_1, a_2, så att $\|\underbrace{e^{-x} - a_1\varphi_1 - a_2\varphi_2}_{v}\|^2$ blir så liten som möjligt.

Minst då $v = P_{[\varphi_1, \varphi_2]}(e^{-x})$.

$$a_k = \frac{(\varphi_k|e^{-x})}{\|\varphi_k\|^2} = \frac{\int_{k-1}^{k} 1 \cdot e^{-x} \cdot e^x dx}{\int_{k-1}^{k} 1 \cdot 1 \cdot e^x dx} = \frac{\int_{k-1}^{k} 1 dx = 1}{\left[e^x\right]_{k-1}^{k}} = \frac{1}{e^k - e^{k-1}} = \frac{1}{e^k\left(1 - \frac{1}{e}\right)} =$$

$$= \frac{e}{e-1} e^{-k}$$

$$\boxed{\|e^{-x}\|^2 = \int_0^\infty e^{-x} e^{-x} e^x dx = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1}$$

minsta värde:

$$\|u\|^2 - \sum_{k=1}^{2} |a_k|^2 \|\varphi_k\|^2 = 1 - \left(\left(\frac{1}{e-1}\right)^2 (e-1)^2 + \left(\frac{1}{e(e-1)}\right)^2 (e(e-1))^2\right) = \dots$$

Kan göra vektorer ortogonala med **Gram-Schmidt**. Om $u_1, \dots, u_n$ linjärt oberoende. Då finns $\varphi_1, \dots, \varphi_n$ parvis ortogonala

$[u_1] = [\varphi_1], [u_1, u_2] = [\varphi_1, \varphi_2] \dots$

$\dots [u_1, u_2, \dots, u_k] = [\varphi_1, \varphi_2, \dots, \varphi_k]$.

$\varphi_1 = u_1$

$\varphi_2 = u_2 - P_{[\varphi_1]} u_2$

$\vdots$

$\varphi_n = u_n - P_{[\varphi_1, \dots, \varphi_{n-1}]} u_n$

Ofta vill man välja ortogonala polynom.

Ex) Ortogonalisera polynomen $u_0(x) = 1$, $u_1(x) = x$, $u_2(x) = x^2$ i skalärprodukten

$$(u|v) = \int_{-1}^{1} u(x) v(x) dx$$

$\varphi_0 = u_0 = 1$

$\|\varphi_0\|^2 = \int_{-1}^{1} 1 \cdot 1 dx = 2.$

$$\varphi_1 = u_1 - P_{[\varphi_0]} u_1 = x - \frac{(1|x)}{\|\varphi_0\|^2} x =$$

$$= x - \frac{\overbrace{\int_{-1}^{1} 1 \cdot x dx}^{=0}}{2} x = x$$

$$\|\varphi_1\|^2 = \int_{-1}^{1} x \cdot x\, dx = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

$$\varphi_2 = u_2 - P_{[\varphi_0, \varphi_1]} u_2 = x^2 - \frac{(\varphi_0|x^2)}{\|\varphi_0\|^2} \varphi_0 - \frac{(\varphi_1|x^2)}{\|\varphi_1\|^2} \varphi_1 = x^2 - \frac{2/3}{2} \cdot 1 = x^2 - \frac{1}{3}$$

$(\varphi_0|x^2) = \int_{-1}^{1} 1 \cdot x^2 dx = \frac{2}{3}$

$(\varphi_1|x^2) = \int_{-1}^{1} x \cdot x^2 dx = 0.$

Vill man ha ON-bas får man normera $\frac{\varphi_0}{\|\varphi_0\|}, \frac{\varphi_1}{\|\varphi_1\|}, \frac{\varphi_2}{\|\varphi_2\|}$ blir $\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$ Väljer istället så att $\frac{\varphi_0}{\varphi_0(1)}, \frac{\varphi_1}{\varphi_1(1)}, \frac{\varphi_2}{\varphi_2(1)}$ – $1, x, \frac{3}{2}(x^2 - 1)$ Kallas Legendrepolynom.

Ex) Bestäm det 2:agradspolynom som bäst approximerar $f(x) = |x|$ på intervallet $[-1,1]$. i normen $\|f\|_2^2 = \int_{-1}^{1} |f(x)|^2 dx$

$L_2([-1,1])$

M = 2:agradspolynom

$(u|v) = \int_{-1}^{1} u(x)v(x)dx$

Har precis funnit ortogonal bas för M, $\varphi_0, \varphi_1, \varphi_2$

Polynomet ges av $P_{[\varphi_0, \varphi_1, \varphi_2]}(|x|) =$

$$= \frac{(\varphi_0 \,|\, |x|)}{\|\varphi_0\|^2}\varphi_0 + \underbrace{\frac{(\varphi_1 \,|\, |x|)}{\|\varphi_1\|^2}}_{0}\varphi_1 + \frac{(\varphi_2 \,|\, |x|)}{\|\varphi_2\|^2}\varphi_2$$

$$= \frac{\int_{-1}^{1} 1\cdot|x|\,dx}{2}\cdot 1 + \frac{\int_{-1}^{1} x\cdot|x|\,dx}{2/3}\cdot x + \cancel{\frac{\int_{-1}^{1} x^2|x|\,dx}{8/45}\left(x^2-\frac{1}{3}\right)}$$

$$+ \frac{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)|x|\,dx}{8/45}\left(x^2-\frac{1}{3}\right) = (\ldots) =$$

$$= \frac{15}{16}x^2 + \frac{3}{16}$$

Norm $\|u\|$ $H \longrightarrow \mathbb{R}$
linjärt rum

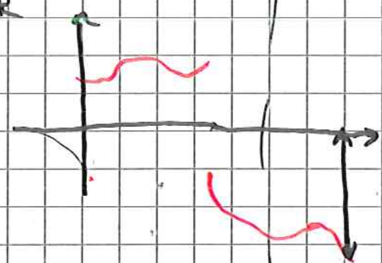
1) $\|u\| \geq 0$, och $\|u\| = 0$ bara om $u = 0$.

2) $\|ku\| = |k|\|u\|$ $k \in \mathbb{R}$

3) $\|u+v\| \leq \|u\| + \|v\|$ triangelolikheten

Ex) Funktionen $u: \Omega \to \mathbb{R}$

$\|u\|_\infty = \sup_{x\in\Omega} |u(x)|$



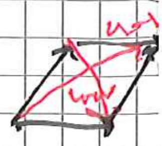
$\|u\|_2 = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}$

Om det finns skalärprodukt $(u|v)$ på H så får vi automatiskt norm

$\|u\| = \sqrt{(u|u)}$

För sådana och bara sådana gäller parallellogramlagen

$\|u+v\|^2 + \|u-v\|^2 =$
$= 2(\|u\|^2 + \|v\|^2)$



$(u+v|u+v) + (u-v|u-v) =$
$= 2(u|u) + 2(v|v)$.

Ex) $\Omega = (0,1)$ $u(x) = 1+x$
$v(x) = 1-x$

Kolla parallellogramlagen i $\|\cdot\|_\infty$

VL $= \|u+v\|_\infty^2 + \|u-v\|_\infty^2 =$
$= \|2\|_\infty^2 + \|2x\|_\infty^2 = 2^2 + 2^2 = 8$.

HL: $2(\|u\|^2 + \|v\|^2) =$
$= 2(\|1+x\|^2 + \|1-x\|^2) =$
$= 2(2^2 + 1^2) = 10$