12/2-2013 Summa, $Z = X + Y$

⑥. Diskret:

$p_Z(k) = P(X+Y=k) = \sum_{i+j=k} P_{X,Y}(i,j) = \sum_{i=0}^{k} P_{X,Y}(i,k-i) = \sum_{i=0}^{k} p_X(i)\cdot p_Y(k-i)$ = Diskret faltning ($p_X * p_Y$)

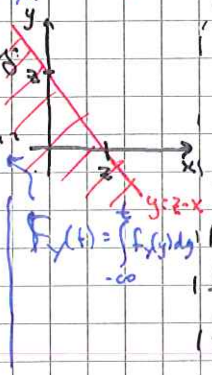
Oberoende

Kontinuerlig:

$F_Z(z) = P(X+Y \le z) = \iint_{x+y\le z} f_{X,Y}(x,y)\,dx\,dy =$ (Ober.)

$= \iint_{x+y\le z} f_X(x)\cdot f_Y(y)\,dx\,dy = \int_{-\infty}^{\infty} f_X(x) \underbrace{\int_{-\infty}^{z-x} f_Y(y)\,dy}_{F_Y(z-x)}\,dx =$

$= \int_{-\infty}^{\infty} f_X(x) F_Y(z-x)\,dx$



$F_Y(t) = \int_{-\infty}^{t} f_Y(y)\,dy$

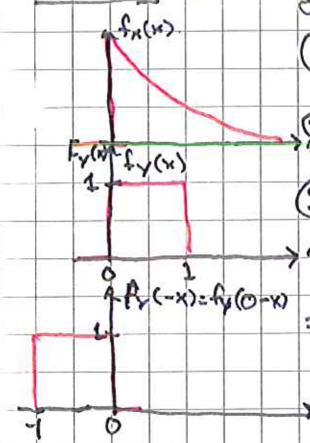
$f_Z(x) = \frac{d}{dz}F(z) = \frac{d}{dz}\int_{-\infty}^{\infty} f_X(x)F_Y(z-x)\,dx =$

$= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$: faltning

Ex) $X \in Exp(1)$, $Y \in U(0,1)$, oberoende:

Bestäm $f_Z(z)$ om $Z = X+Y$

Alt 1: faltning $f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\cdot f_Y(z-x)\,dx$.

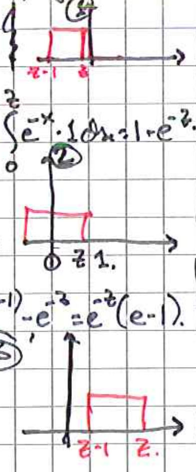


① $\boxed{z<0}$ $f_Z(z) = 0$

② $\boxed{0 \le z \le 1}$ $f_Z(z) = \int_0^z e^{-x}\cdot 1\,dx = 1 - e^{-z}$.

③ $\boxed{z>1}$ $f_Z(z) = \int_{z-1}^{z} e^{-x}\cdot 1\,dx =$

$= \left[-e^{-x}\right]_{z-1}^{z} = e^{-(z-1)} - e^{-z} = e^{-z}(e-1)$.



$$f_Z(z) = \begin{cases} 0 & z<0 \\ 1-e^{-z} & 0 \le z \le 1 \\ e^{-z}(e-1), & z>1. \end{cases}$$

Alt: Dubbelintegral

$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = e^{-x}\cdot 1$, $x \ge 0$, $0 \le y \le 1$.

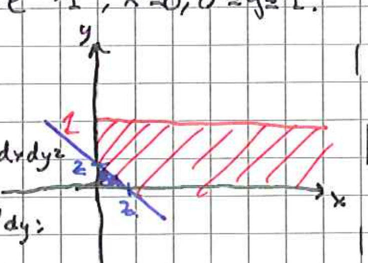
$\boxed{z<0}$ $f_Z(z) = 0$

$\boxed{0 \le z \le 1}$ $F_Z(z) = \iint_{x+y\le z} f_{X,Y}(x,y)\,dx\,dy =$

$= \int_{y=0}^{z}\int_{x=0}^{z-y} e^{-x}\,dx\,dy = \int_0^z \left[-e^{-x}\right]_0^{z-y} dy =$

$= \int_0^z 1 - e^{-(z-y)}\,dy = \left[y - e^{-(z-y)}\right]_0^z = z - e^{-(z-z)} - (0 - e^{-z}) = z - 1 + e^{-z}$

$f_Z(z) = 1 - e^{-z}$ $0 \le z \le 1$



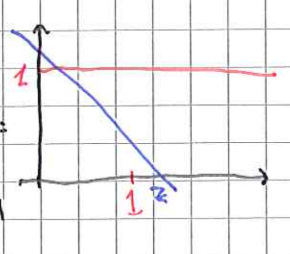
$\boxed{z>1}$ $F_Z(z) = \int_{y=0}^{1}\int_{x=0}^{z-y} e^{-x}\,dx\,dy =$

$= \left[y - e^{-(z-y)}\right]_0^1 = 1 - e^{-(z-1)} - (0 - e^{-z}) =$

$= 1 - e^{-(z-1)} + e^{-z}$

$f_Z(z) = e^{-(z-1)} - e^{-z} = e^{-z}(e-1)$ $z>1$

$\frac{d}{dz}F_Z(z)$



Ex) Vad får man i medeltal för utfall på en tärning? Jo, $\frac{1+2+3+4+5+6}{6} = 3.5$.

Ifr $x_t = \frac{\int x f(x)\,dx}{\int f(x)\,dx}$ "Tyngdpunkten". $\int f(x)\,dx = 1$ i vårt fall.

Ex) Keno-3. X = Antal vinster. Y = Vinst i kr

k	0	1	2	3
$P_X(k)$	0.38	0.45	0.17	0.02.

k'	0	5	90
$P_Y(k')$	0.81	0.17	0.02

Hur mycket vinner man i medeltal?

Jo $-E(Y) = \sum_k k\cdot P_Y(k) = 0 + 0.81 + 5\cdot 0.17 + 90\cdot 0.02 = 2.65$

Nettovinst $= E(Y) - 5 = \underline{-2.35}$.

Ex)

Linjär funktion $Y = g(X) = a\cdot X + b$...

$E(Y) = E(g(X)) = E(aX+b) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \overset{g(x)=ax+b}{=} \int_{-\infty}^{\infty}(ax+b)f_X(x)\,dx =$

$= a\underbrace{\int_{-\infty}^{\infty} x f_X(x)\,dx}_{E(X)} + b\underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{=1} = aE(x) + b.$, dvs $E(aX+b) = aE(x) + b$

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Summa av två oberoende, $Z = X + Y$

Diskret

$$p_Z(k) = \sum_{i+j=k} p_X(i) \cdot p_Y(j) = \sum_{i=0}^{k} p_X(i) p_Y(k-i)$$

Diskret faltning $(p_X * p_Y)$

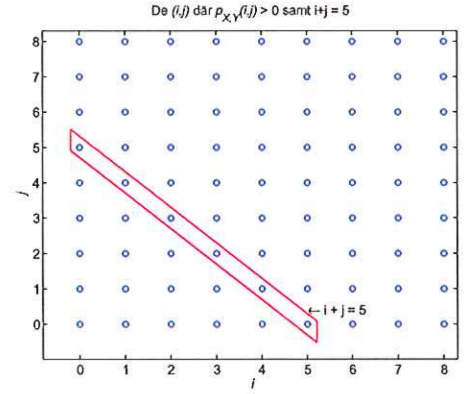
Kontinuerlig

$$F_Z(z) = \iint_{x+y \le z} f_X(x) \cdot f_Y(y)\,dx dy = \int_{-\infty}^{\infty} f_X(x) F_Y(z-x)\,dx$$

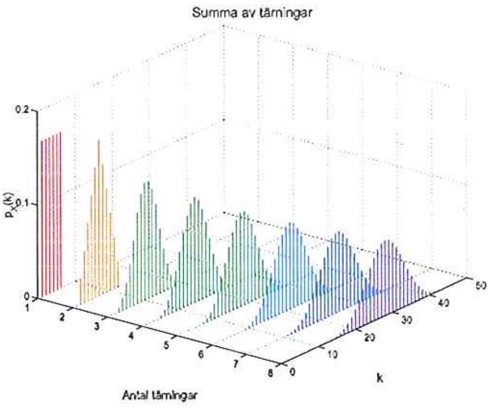
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

F5 − 1

Område i (i,j)-planet där koordinatsumman är konstant

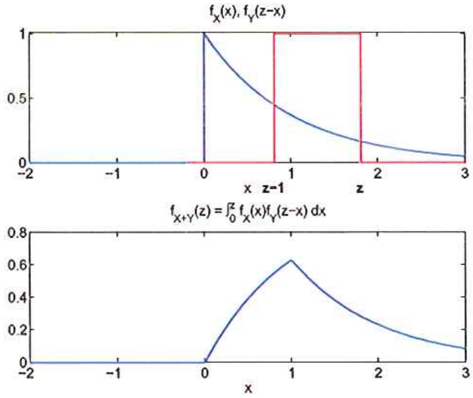


F5 − 2



F5 − 3

Faltning



F5 − 4

Störst av två oberoende $Z = \max(X, Y)$

$$F_Z(z) = P(Z \le z) = P(\max(X,Y) \le z) = P(X \le z \cap Y \le z) = F_X(z) F_Y(z)$$

Störst av fler oberoende $Z = \max(X_1, \ldots, X_n)$

$$F_Z(z) = F_{X_1}(z) \cdot \ldots \cdot F_{X_n}(z)$$

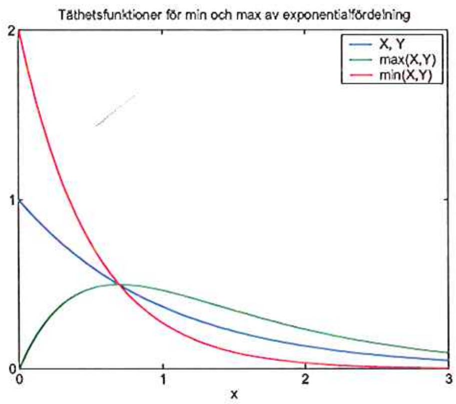
Minst av två oberoende $Z = \min(X, Y)$

$$F_Z(z) = P(Z \le z) = P(\min(X,Y) \le z) = 1 - P(\min(X,Y) > z) = 1 - P(X > z \cap Y > z) = 1 - [1 - F_X(z)][1 - F_Y(z)]$$

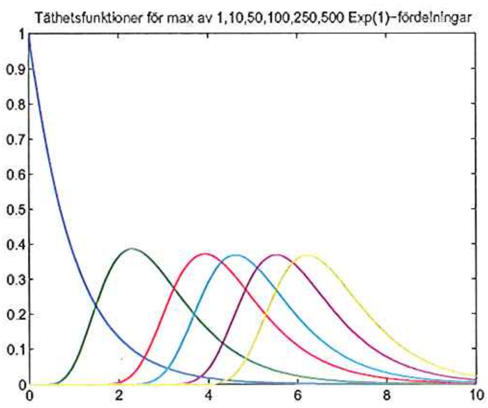
Minst av fler oberoende $Z = \min(X_1, \ldots, X_n)$

$$F_Z(z) = 1 - [1 - F_{X_1}(z)] \cdot \ldots \cdot [1 - F_{X_n}(z)]$$

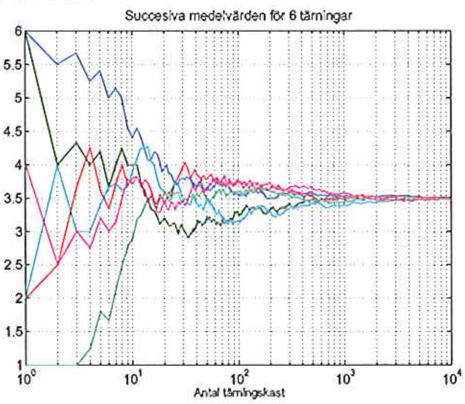
F5 − 5



F5 − 6



Väntevärden



Lägesmått

Väntevärde, $E(X), \mu, \mu_X, m, \ldots$

$$E(X) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} x f_X(x)\,dx & \text{Kont.} \\ \displaystyle\sum_k k p_X(k) & \text{Diskr.} \end{cases}$$

Väntevärdet anger *tyngdpunkten* för fördelningen och kan tolkas som det värde man får i "*medeltal* i långa loppet".

Väntevärde av $Y = g(X)$

$$E(Y) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} g(x) f_X(x)\,dx & \text{Kont.} \\ \displaystyle\sum_k g(k) p_X(k) & \text{Diskr.} \end{cases}$$

Spridningsmått

Varians, $V(X), \sigma^2, \sigma_X^2$

$$V(X) = E\{[X - E(X)]^2\} = E(X^2) - E(X)^2$$

- Variansen anger hur utspridd X är kring sitt väntevärde.
- Variansen är alltid positiv.

Standardavvikelse, $D(X), \sigma, \sigma_X$

$$D(X) = \sqrt{V(X)}$$

- Standardavvikelsen har samma dimension som X och $E(X)$.