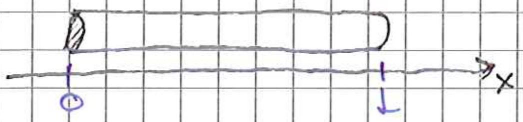
25/1 2012 (4)

Lösning värmeledningsekv.

1-rumsdimension:

Ex. Metallstav, längd L, värmd till 100°C, isolerad utom i ändarna.
Stoppas i isvatten. Beräkna temperaturen fördelad i staven.

Modell:
Temperatur $u(x,t)$.



$$\begin{cases} \partial_t u - a\underbrace{\partial_x^2 u}_{=\Delta u} & a: \text{värmeledningskoefficient} \\ u(0,t)=0 & 0<x<L \\ u(L,t)=0 & t>0 \\ u(x,0)=100°C & 0<x<100 \end{cases}$$

Kan förenkla ekvationen genom enkla variabelbyten (s.03)

$$y=\frac{\pi}{L}x \Rightarrow \begin{cases} x=0 \Leftrightarrow y=0 \\ x=L \Leftrightarrow y=\pi \end{cases}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}\frac{\partial y}{\partial x}=\frac{\partial u}{\partial y}\frac{\pi}{L}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\pi}{L}\frac{\partial u}{\partial y}\right)=\frac{\pi^2}{L^2}\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \left[\partial_t u - a\frac{\pi^2}{L^2}\partial_y^2 u = 0\right].$$

$$s=\frac{a\pi^2}{L^2}t,\quad \frac{\partial u}{\partial t}=\frac{\partial u}{\partial s}\frac{\partial s}{\partial t}=\frac{a\pi^2}{L^2}\frac{\partial u}{\partial s}$$

$$\Rightarrow \partial_t u - \frac{a\pi^2}{L^2}\partial_y^2 u = 0 = \frac{a\pi^2}{L^2}\partial_s u - \frac{a\pi^2}{L^2}\partial_y^2 u = 0$$

$$\Rightarrow \partial_s u - \partial_y^2 u = 0.$$

∴ Ny modell:

$$\begin{cases} (1)\ \partial_s u - \partial_y^2 u = 0 & 0<y<\pi,\ s>0 \\ (2)\ u(0,t)=0,\ u(\pi,t)=0 & s>0 \\ (3)\ u(y,0)=100 & 0<y<\pi \end{cases}$$

Sök först speciella (enkla) lösningar, som kan skrivas $u(y,s)=Y(y)\,S(s)$
Y, S är icke-triviala ($\neq 0$) funktioner av 1 variabel. ($u=0$ löser 1,2 men ej 3.)

$$u'_s - u''_{yy} = Y(y)S'(s) - Y''(y)S(s) = 0 \quad (\forall u = Y(y)S(s))$$

$$\Leftrightarrow Y(y)S'(s) = Y''(y)S(s) \Leftrightarrow \frac{S'(s)}{S(s)} = \frac{Y''(y)}{Y(y)} = \boxed{-\lambda}$$

konstant.

Metoden kallas att separera variabler.

$u = Y(y)S(s)$ löser (1).

$$\begin{cases} Y''(y)+\lambda Y(y)=0 & (4) \text{ för något } \lambda \\ S'(s)+\lambda S(s)=0 & (5) \end{cases}$$

$S(s)=Ce^{-\lambda s}$ för $C\neq 0$.

(4) Löser vi med (2): $u(0,s)=Y(0)S(s)=0$
$\Rightarrow Y(0)=0$
$u(\pi,s)=Y(\pi)S(s)=0 \Rightarrow Y(\pi)=0.$

$$\begin{cases} Y''+\lambda Y=0 \\ Y(0)=0,\ Y(\pi)=0 \end{cases}$$

Om $\boxed{\lambda<0}$: Sätt $\lambda=-\omega^2$
Karakteristisk ekvation:
$r^2-\omega^2=0$
$r=\pm\omega$
$Ae^{\omega y}+Be^{-\omega y}$
$Y(0)=A+B=0 \Rightarrow B=-A.$
$Y(\pi)=Ae^{\omega\pi}+Be^{-\omega\pi}=A(\underbrace{e^{\omega\pi}-e^{-\omega\pi}}_{>0})=0$

$\Rightarrow A=B=0$ (Trivial lösning)

$\boxed{\lambda=0}$: $Y''=0$ $Y(y)=Ay+B$
$Y(0)=B=0$ $B=0$
$Y(\pi)=A\pi+0=A\pi \Rightarrow A=0$
(Åter, trivial lösning...)

$\boxed{\lambda>0}$: $\lambda=\omega^2$; $r^2+\omega^2=0 \Rightarrow r=\pm i\omega$

$Y(y)=A\sin(\omega y)+B\cos(\omega y)$
$Y(0)=A\cdot 0+B=B=0$
$Y(\pi)=A\sin(\omega\pi)=0.$
$\Rightarrow A\neq 0$ om $\sin(\omega\pi)=0$
$\Rightarrow \omega=k,\ k=1,2,3\ldots \Rightarrow \lambda=k^2.$
Om $\lambda=k^2$, $k=1,2,3\ldots$ så är
$u_k=C_k e^{-k^2 s}\sin(ky)$ (Den icke-triviala lösningen)

$u_k(y,0)=C_k\sin(ky)$
$u(y,s)=\sum_{k=1}^{\infty}C_k e^{-k^2 s}\sin(ky)$; löser (1) och (2).
$u(y,0)=\sum_{k=1}^{\infty}C_k\cdot 1\cdot\sin(ky)=100$
$C_k=b_k \Rightarrow \boxed{u(y,s)=\sum_{k=1}^{\infty}b_k e^{-k^2 s}\sin(ky)}$

Sinusutveckling:

$$b_k=\frac{2}{\pi}\int_0^{\pi}100\sin(ky)\,dy=\frac{200}{\pi}\left[-\frac{\cos(ky)}{k}\right]_0^{\pi}=$$

$$=\frac{200}{\pi}\frac{1}{k}(1-\underbrace{\cos(k\pi)}_{(-1)^k})=\frac{200}{\pi}\frac{1}{k}\begin{bmatrix}2 & k=1,3,5,\ldots \\ 0 & k=2,4,6,\ldots\end{bmatrix}$$

Löst! $u(y,s)=\sum_{k=1}^{\infty}\frac{200}{\pi}\frac{1}{k}(1-(-1)^k)e^{-k^2 s}\sin(ky)$
Inför var.bytena igen:

$$u(x,t)=\sum_{k=1}^{\infty}\frac{200}{\pi}\frac{1-(-1)^k}{k}e^{-\frac{a\pi^2}{L^2}k^2 t}\sin\left(\frac{k\pi}{L}x\right)$$

$$=\sum_{i=0}^{\infty}\frac{400}{\pi(2i+1)}e^{-\frac{a\pi^2}{L^2}(2i+1)^2 t}\sin\left(\frac{(2i+1)}{L}\pi x\right) \quad \{k=2i+1,\ i\geq 0\}$$

Ledande termer: 1

Kan ansätta sinusutveckling direkt:
$u(y,s)=\sum b_k(s)\sin(ky)$