30/1 – 2012

Avsnitt 7 i kursprogrammet är struket i år.

3. Trippelintegral (Rep.)

$$\iiint_B f(x,y,z)\,dx\,dy\,dz$$

Linjeintegral (kurv-)

$$\int_\gamma \bar{F}\cdot d\bar{r} \qquad \underset{\text{(sluten)}}{\oint_\gamma} \bar{F}\cdot d\bar{r}. \quad \text{om } \bar{F} = \nabla U(x,y,z)$$

$$\text{där } \int_\gamma \bar{F}\cdot d\bar{r} = U(\text{slut}) - U(\text{start})$$

Ytintegral (flödesintegral)

$$\iint_D \bar{F}\cdot \underbrace{\hat{n}\,dS}_{d\bar{S}} \qquad \underset{\text{sluten yta}}{\oint\!\!\oint_D} \bar{F}\cdot d\bar{S}$$

Ex) $\bar{F} = \begin{bmatrix} 2x \\ y \\ x^2 \end{bmatrix}$ yta: sfär runt origo med radien R.

$$\bar{r}(u,v) = \begin{bmatrix} R\cos u\sin v \\ R\sin u\sin v \\ R\cos v \end{bmatrix}, \quad D: \begin{Bmatrix} 0 < u < 2\pi \\ 0 < v < \pi \end{Bmatrix}$$

$$\oint\!\!\oint_D \bar{F}\cdot\hat{n}\,dS = \iint_{(u,v)\in D} \bar{F}(\bar{r}(u,v))\cdot \pm\left(\frac{d\bar{r}}{du}\times\frac{d\bar{r}}{dv}\right)du\,dv$$

$$= (\ldots) = \int_0^{2\pi}\int_0^{\pi} \begin{bmatrix} 2R\cos u\sin v \\ R\sin u\sin v \\ R^2\cos^2 u\sin^2 v \end{bmatrix}\cdot\begin{bmatrix} \cos u\sin v \\ \sin u\sin v \\ \cos v \end{bmatrix} R^2\sin v\,du\,dv$$

$$= (\ldots)$$

$$\hat{n} = \frac{1}{R}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\oint\!\!\oint_D \begin{bmatrix} 2x \\ y \\ x^2 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}\frac{1}{R}\,dS = \oint\!\!\oint_D (2x^2 + y^2 + \underset{=0}{x^2 z})\frac{1}{R}\,dS =$$

$$= \oint\!\!\oint_D (2x^2+y^2)\frac{1}{R}\,dS = \left|\ \underset{\text{sfär runt origo}}{\oint\!\!\oint} x^2\,dS = \oint\!\!\oint y^2\,dS = \oint\!\!\oint z^2\,dS \ \right|$$

$$= \frac{1}{R}\oint\!\!\oint_D \underbrace{(x^2+y^2+z^2)}_{R^2}\,dS = \frac{R^2}{R}\oint\!\!\oint_D dS = R\cdot\begin{bmatrix}\text{arean av en sfär med radie } R\end{bmatrix}$$

$$= 4\pi R^3$$

GAUSS SATS / DIVERGENSSATSEN

$$\nabla\cdot\bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

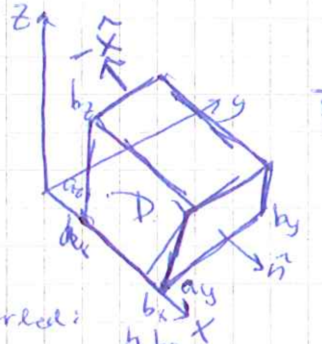
$$\oint\!\!\oint_{\partial D} \bar{F}\cdot\hat{n}\,dS = \iiint_D (\nabla\cdot\bar{F})\,dV$$

flödet ut genom ytan = summan av produktionen

Ex) $\oint\!\!\oint_D \bar{F}\cdot\hat{n}\,dS = \iiint_{V_D} (\nabla\cdot\bar{F})\,dV = \iiint_{V_D} (2+1)\,dV =$

$$= 3\iiint_{V_D} dV = \cancel{3}\frac{4\pi R^3}{\cancel{3}} = 4\pi R^3$$

Bevis:



$$\bar{F} = \begin{bmatrix} F_x \\ 0 \\ 0 \end{bmatrix}$$

Vänsterled:

$$\oint\!\!\oint_{\partial D} \bar{F}\cdot\hat{n}\,dS = \int_{a_y}^{b_y}\int_{a_z}^{b_z} F_x(b_x,y,z)\overbrace{\hat{x}\cdot\hat{x}}^{1}\,dy\,dz +$$

$$+ \int_{a_y}^{b_y}\int_{a_z}^{b_z} F_x(a_x,y,z)\underbrace{\hat{x}\cdot(-\hat{x})}_{-1}\,dy\,dz =$$

$$= \int_{a_y}^{b_y}\int_{a_z}^{b_z} \left(F_x(b_x,y,z) - F_x(a_x,y,z)\right)dy\,dz$$

Högerled:

$$\iiint_D (\nabla\cdot\bar{F})\,dV = \int_{a_z}^{b_z}\int_{a_y}^{b_y}\int_{a_x}^{b_x}\left(\frac{\partial F_x(x,y,z)}{\partial x}\right)dx\,dy\,dz =$$

$$= \int_{a_z}^{b_z}\int_{a_y}^{b_y}\left[F_x(b_x,y,z) - F_x(a_x,y,z)\right]dy\,dz$$

HL = VL