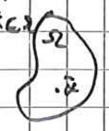
(20) 18/4 2012

Repl:

Greenfunktion $G(\bar{x},\bar{\alpha})$ löser

$$\begin{cases} -\Delta G(\bar{x},\bar{\alpha}) = \delta_{\bar{\alpha}}(\bar{x}) & \bar{x}\in\Omega \\ G = 0 & \bar{x}\in\partial\Omega \end{cases}$$



$$\int_\Omega \delta_{\bar{\alpha}}(\bar{x})\varphi(\bar{x})\,d\bar{x} = \varphi(\bar{\alpha})$$

Huvudsats för Greenfunktionen

$$\begin{cases} -\Delta u = f & i\ \Omega \\ u = g & på\ \partial\Omega \end{cases}$$

har lösning

$$u(\bar{x}) = \int_\Omega G(\bar{x},\bar{\alpha})f(\bar{\alpha})\,d\bar{\alpha} - \int_{\partial\Omega} \partial_{n_\alpha} G(\bar{x},\bar{\alpha})g(\bar{\alpha})\,dS_{\alpha}$$

Bevis: Green II

$$\int_\Omega u(\bar{\alpha})\underbrace{\Delta_\alpha G(\bar{x},\bar{\alpha})}_{\Delta_\alpha G(\bar{x},\alpha) = \delta_x(\bar{\alpha})} - G(\bar{x},\bar{\alpha})\underbrace{\Delta_\alpha u(\alpha)}_{=f(\bar{\alpha})}\,d\bar{\alpha} = \int_{\partial\Omega}$$

$$= \int_{\partial\Omega} \underbrace{u(\bar{\alpha})}_{=g(\bar{\alpha})}\partial_{n_\alpha} G(\bar{x},\bar{\alpha}) - \underbrace{G(\bar{x},\bar{\alpha})}_{=G(\bar{\alpha},\bar{x})=0}\partial_{n_\alpha} u(\bar{\alpha})\,dS_\alpha$$

$$= \int \partial_{n_\alpha} G(\bar{x},\bar{\alpha})g(\bar{\alpha})\,dS_{\bar{\alpha}} = -u(\bar{x}) + \int_\Omega G(\bar{x},\bar{\alpha})f(\bar{\alpha})\,d\bar{\alpha}$$

Konstatera Greenfunktionen utgående från fundamentallösningar K för $\mathbb{R}^n$.

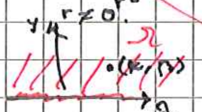
$$-\Delta K = \delta$$

$$K(\bar{x}) = \begin{cases} -x\,\theta(x) & \mathbb{R} \\ -\frac{1}{2\pi}\ln|\bar{x}| & i\ \mathbb{R}^2 \\ -\frac{1}{4\pi}\frac{1}{|x|} & i\ \mathbb{R}^3 \end{cases}$$

$\boxed{\text{Koll:}}$ i $\mathbb{R}^3$, polära koord $-\Delta K = -\frac{1}{r^2}\partial_r(r^2\partial_r K)$

$$-\frac{1}{r^2}\Delta K = 0 \quad r\neq 0$$

Ex Ω övre halvplan

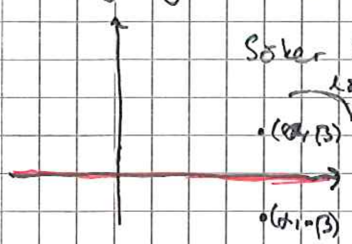


Söker Greenfunktionen

$$-\Delta G = \delta_{(\alpha,\beta)}(x,y) \quad : (-\partial_x^2 - \partial_y^2)G(x,y,\alpha,\beta)$$

$$G(x,0;\alpha,\beta) = 0$$

Speglingstrick



Söker G udda i y-led.

Lös $-\Delta G = \delta_{(\alpha,\beta)} - \delta_{(\alpha,-\beta)}$ i $\mathbb{R}^2$

Addera fundamentallösningar (translaterade)

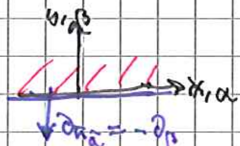
$$G(x,y,\alpha,\beta) = -\frac{1}{2\pi}\ln|(x,y)-(\alpha,\beta)| + \frac{1}{2\pi}\ln|(x,y)-(\alpha,-\beta)| =$$

$$= \frac{1}{4\pi}\left(\ln((x-\alpha)^2 + (y-\beta)^2) - \ln((x-\alpha)^2 + (y+\beta)^2)\right)$$

Koll $G(x,0;\alpha,\beta) = \cancel{\frac{1}{4\pi}}(\ ) = 0$

Ex Lös m.h.a. huvudsatsen

$$\begin{cases} -\Delta u = 0 & x\in\mathbb{R},\ y>0 \\ u(x,0) = g(x) & x\in\mathbb{R} \end{cases}$$



$$u(x,y) = -\int_{-\infty}^{\infty} -\partial_\beta G(x,y,\alpha,\beta)\Big|_{\beta=0} g(\alpha)\,d\alpha$$

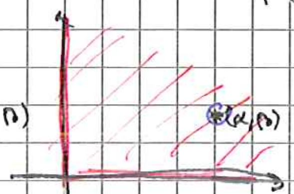
$$= \int_{-\infty}^{\infty} \frac{1}{4\pi}\left(\frac{2(\beta-y)}{(x-\alpha)^2+(y-\beta)^2} - \frac{2(y+\beta)}{(x-\alpha)^2+(y+\beta)^2}\right)\Bigg|_{\beta=0} g(\alpha)\,d\alpha$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\alpha)^2+y^2}\,g(\alpha)\,d\alpha = \int_{-\infty}^{\infty} P(x-\alpha,y)g(\alpha)\,d\alpha$$

Poissons kärna

Ex Konstruera en Greenfunktion m.h.a. speglingar för

$$\begin{cases} -\Delta G = \delta_{(\alpha,\beta)} \\ G(x,0;\alpha,\beta) = 0 & x>0 \\ G(0,y;\alpha,\beta) = 0 & y>0 \end{cases}$$



Spegla udda i både x- och y-axeln.

$$-\Delta G = \delta_{(\alpha,\beta)} - \delta_{(-\alpha,\beta)} - \delta_{(\alpha,-\beta)} + \delta_{(-\alpha,-\beta)}$$

$$G(x,y;\alpha,\beta) = -\frac{1}{2\pi}\ln|(x,y)-(\alpha,\beta)| + \frac{1}{2\pi}\ln|(x,y)-(-\alpha,\beta)| + \frac{1}{2\pi}\ln|(x,y)-(\alpha,-\beta)| - \frac{1}{2\pi}\ln|(x,y)-(-\alpha,-\beta)|$$

$$= \frac{1}{4\pi}\Big(\ln((x-\alpha)^2+(y+\beta)^2) - \ln((x+\alpha)^2+(y-\beta)^2) - \ln((x-\alpha)^2+(y+\beta)^2) + \ln((x+\alpha)^2+(y+\beta)^2)\Big)$$

Ex Lös

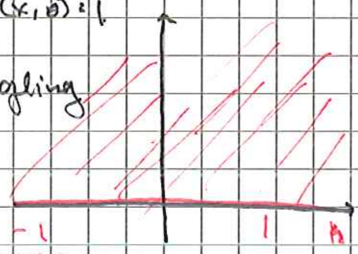
$$\begin{cases} -\Delta u = \delta_{(\alpha,\beta)} \\ u(0,y) = 0 \\ u(x,0) = 1 \end{cases}$$

Dela upp i 2 delar:

① $\begin{cases} -\Delta u = \delta_{(\alpha,\beta)} \\ u(0,y) = 0 \\ u(x,0) = 0 \end{cases}$ ② $\begin{cases} -\Delta u = 0 \\ u(0,y) = 0 \\ u(x,0) = 1 \end{cases}$

① har vi precis löst!

② Kan vi förenkla med spegling udda map y-axeln



$$\begin{cases} -\Delta u = 0 \\ u(x,0) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases} = g(x) = \operatorname{sgn}(x) \end{cases}$$

Halvplan!

Löses med Poissonkärna

$$u(x,y) = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{y}{(x-\alpha)^2+y^2}\operatorname{sgn}(\alpha)\,d\alpha$$

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{y}{(x-\alpha)^2+y^2}(-1)\,d\alpha + \int_0^\infty \frac{y}{(x-\alpha)^2+y^2}\,1\,d\alpha =$$

$$= \frac{1}{\pi}\int_{-\infty}^{0} \frac{1}{y}\frac{1}{1+\left(\frac{\alpha-x}{y}\right)^2}\,d\alpha$$

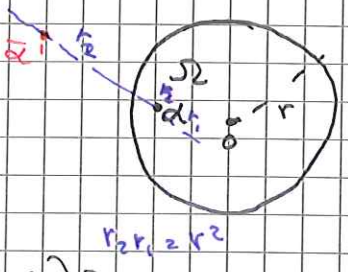
$$\frac{1}{\pi}\left(\left[-\arctan\left(\frac{\alpha-x}{y}\right)\right]_{-\infty}^{0} + \left[\arctan\left(\frac{\alpha-x}{y}\right)\right]_0^\infty\right) =$$

$$= \frac{1}{\pi}\left(-\arctan\left(-\frac{x}{y}\right) - \frac{\pi}{2} + \frac{\pi}{2} - \arctan\left(-\frac{x}{y}\right)\right) =$$

$$= \frac{2}{\pi}\arctan\left(\frac{x}{y}\right).$$

Greenfunktion $G(\bar{x}, \bar{a})$ till cirkelskiva Ω med radie r.

$$\begin{cases} -\Delta G = \delta_{\bar{\alpha}} & \text{i } \Omega \\ G = 0 & \text{på } \partial\Omega \end{cases}$$



$$|x - \alpha| \equiv \frac{|x|}{r} |x - \alpha'| \text{ då } x \in \partial\Omega$$

$$G(\bar{x}, \bar{\alpha}) = -\frac{1}{2\pi}\left(\ln|x - \alpha| - \ln\frac{|x|}{r}|x - \alpha'|\right)$$