(19) Rep: DATORÖVN 3. MÅN 23/4 8-10 E:Uranus

17/4 2012

Transformmetod

Ex1 Laplace ekv + Dirichletvillkor för halvplan

$$\begin{cases} -\Delta u = 0 \\ u(x,0) = g(x) \\ u \text{ begränsad} \end{cases}$$

Kommentar:

Ex1 $\begin{cases} -\Delta u = 0 \\ u(x,0) = 0 \end{cases}$

$u \equiv 0$ är en lösning, finns fler, t.ex. x-oberoende lösning. $u(x,y) = v(y)$

$$\begin{cases} -v'' = 0 & v'(y) = A & v(y) = Ay + B \\ & & v(0) = B = 0 \\ v(0) = 0 & & v(y) = Ay \end{cases}$$

Åter Fouriertransformerna i x-led

$0 = \hat{0} = -\widehat{\Delta u} = -\widehat{\partial_x^2 u} - \widehat{\partial_y^2 u} = -\partial_x^2 \hat{u} - \partial_y^2 \hat{u} =$

$= -(i\xi)^2 \hat{u} - \partial_y^2 \hat{u} = -\partial_y^2 \hat{u} - \xi^2 \hat{u}$

$r^2 - \xi^2 = 0 \quad r = \pm\xi$

$\hat{u}(\xi, y) = a(\xi) e^{\xi y} + b(\xi) e^{-\xi y}$

$\hat{u}(\xi, 0) = \hat{g}(\xi) \qquad \hat{u}(\xi, 0) = a(\xi) + b(\xi) = \hat{g}(\xi)$

u snäll, $\hat{u}$ snäll $\quad a(\xi) = 0$ då $\xi > 0$

$b(\xi) = 0 \qquad \xi < 0$

$\hat{u}(\xi, y) = \hat{g}(\xi) e^{-y|\xi|}$

$e^{-|\xi|} \longrightarrow \dfrac{2}{1+x^2}$

$e^{-y|\xi|} = e^{-|y\xi|} \longrightarrow \dfrac{2}{|y|(1+(\frac{x}{y})^2)} = \dfrac{2y^2}{y(y^2+x^2)} = \dfrac{2y}{y^2+x^2} \longrightarrow$

$\longrightarrow 2\pi e^{-y|\xi|} \qquad P(x,y) = \dfrac{1}{\pi} \dfrac{y}{x^2+y^2} \longrightarrow e^{-y|\xi|}$

Poissonkärnan. $\hat{u}(\xi, y) = \hat{g}(\xi) \widehat{\dfrac{1}{\pi} \dfrac{y}{x^2+y^2}}(\xi) =$

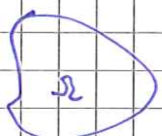
$= \hat{g} \hat{P} = \widehat{g * P}$

$u(x,y) = g * P = \int P(x-t, y) g(t)\, dt = \dfrac{1}{\pi} \int_{-\infty}^{\infty} \dfrac{y}{(x-t)^2 + y^2} g(t)\, dt$

OBS Om $g(x) = \delta(x)$ så blir $u(x,y) = \delta * P =$
$= P = \dfrac{1}{\pi} \dfrac{y}{x^2+y^2}$

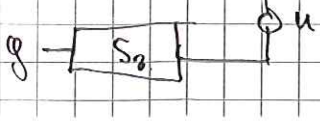
Mer allmänt:

$$\begin{cases} -\Delta u = f & i\ \Omega \\ u = g & \text{på } \partial\Omega \end{cases}$$



kan delas upp:

(1) $\begin{cases} -\Delta u = f & i\ \Omega \\ u = 0 & \text{på } \partial\Omega \end{cases}$ (2) $\begin{cases} -\Delta u = 0 & i\ \Omega \\ u = g & \text{på } \partial\Omega \end{cases}$



för att studera $f \to \boxed{S} \to u$

räcker det att lösa $\partial_\alpha \to \boxed{} \to k(x;\alpha)$

$u(x) = \int_{-\infty}^{\infty} k(x,\alpha) f(\alpha)\, d\alpha$

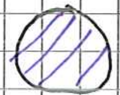
Om tids/translationsinvarians räcker det att känna till $\delta_0(x)$, $\alpha = 0$

$u(x) = \int_{-\infty}^{\infty} \underbrace{k(x-\alpha, 0)}_{h(x-\alpha)} f(\alpha)\, d\alpha = h * f$

Ex1

$$\begin{cases} -\Delta u = 0 & \text{på enhetscirkeln } x^2+y^2<1 \\ u = g & \text{på } \partial\Omega \end{cases}$$

$r = 1$.



$u(1, \theta) = g(\theta)$

translationsinvariant i θ-led

$$\begin{cases} -\frac{1}{r}\partial_r(r \partial_r u) - \frac{1}{r^2}\partial_\theta^2 u = 0 \\ u(1,\theta) = \delta(\theta) \end{cases}$$

$u(r,\theta) = \sum_{-\infty}^{\infty} u_k(r) e^{ik\theta}$; insatt i ekvationen

$\sum_{-\infty}^{\infty} \underbrace{\left(u_k'' + \frac{1}{r} u_k' - \frac{k^2}{r^2} u_k\right)}_{=0} e^{ik\theta} = 0$

$$u_k(r) = \begin{cases} a_k r^k + b_k r^{-k} & k \neq 0 \\ a_0 + b_0 \ln r & k = 0 \end{cases}$$

$u(1) = \sum u_k(1) e^{ik\theta}$.

$$\begin{cases} a_k + b_k = \frac{1}{2\pi} \\ a_0 = \frac{1}{2\pi} \end{cases} \qquad u_k(1) = \frac{(e^{ik\theta}|\delta)}{(e^{ik\theta}|e^{ik\theta})} = \frac{1}{2\pi}$$

u begr då $r \to 0$

$u(r,\theta) = \dfrac{1}{2\pi}\left(\sum_{k=-\infty}^{-1} r^{-k} e^{ik\theta} + 1 + \sum_{k=1}^{\infty} r^k e^{ik\theta}\right) =$

$= \dfrac{1}{2\pi} \sum_{k=1}^{\infty} (re^{-i\theta})^k + 1 + \sum_{k=1}^{\infty} (re^{i\theta})^k =$

$= \dfrac{1}{2\pi}\left(\dfrac{re^{-i\theta}}{1-re^{-i\theta}} + 1 + \dfrac{re^{i\theta}}{1-re^{i\theta}}\right) =$

$= \ldots = \dfrac{1}{2\pi} \dfrac{1-r^2}{1+r^2-2r\cos\theta} = P(r,\theta)$

$$\begin{cases} -\Delta u = 0 & i\ \Omega \\ u = g & \text{på } \partial\Omega \end{cases}$$

har lösningen

$u(r,\theta) = \int_{-\pi}^{\pi} P(r, \theta-\alpha) g(\alpha)\, d\alpha =$

$= \dfrac{1}{2\pi} \int_{-\pi}^{\pi} \dfrac{1-r^2}{1+r^2-2r\cos(\theta-\alpha)} g(\alpha)\, d\alpha$

I hela rummet ($\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$)

Ekv $-\Delta u = f$

$-\Delta$ translationsinvariant, söker fundamentallösning K

$-\Delta K = \delta$

då är $u = K * f$ lösning till givna problemet, ty

$$-\Delta u = -\Delta(K*f) = (-\Delta K)*f = \delta * f = f.$$

SATS $-\Delta$ har fundamentallösningen

$$K = \begin{cases} -x\,\Theta(x) & i\ \mathbb{R} \\ -\frac{1}{2\pi}\ln|\bar{x}| & i\ \mathbb{R}^2 \\ \frac{1}{4\pi}\frac{1}{|\bar{x}|} & i\ \mathbb{R}^3 \end{cases}$$

$|\bar{x}| = \sqrt{x_1^2+x_2^2+x_3^2}$

$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$

Bevis: i $\mathbb{R}$

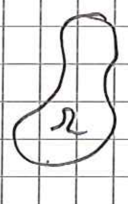
$-K'' = \delta \qquad K' = -\Theta + A \qquad K = -x\Theta + Ax + B$

A, B godtyckliga $\Rightarrow A, B = 0.$

$$-\Delta u = f \sim u(\bar{x}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{1}{|\bar{x}-\bar{y}|} f(\bar{y})\, d\bar{y}$$

Def. Lösningen till

$$\begin{cases} -\Delta u = \delta_\alpha(\bar{x}) & i\ \Omega \\ u(\bar{x}) = 0 & på\ \partial\Omega \end{cases}$$



kallas $\boxed{\text{Greenfunktion}}$ för Dirichletproblem.

$G(x, \alpha)$

SATS G har symmetriegenskap $G(x,\alpha) = G(\alpha, x)$

Bevis

Green II $\displaystyle\int_\Omega u\Delta v - v\Delta u\, dx = \int_{\partial\Omega} u\partial_n v - v\partial_n u\, dS$

Sätt in $u = G(\bar{x}, \alpha)$ $v = G(x, \beta)$ $\Rightarrow -G(\beta,\alpha) + G(\alpha,\beta) =$

$$= \int_\Omega G(\bar{x},\alpha)\underbrace{\Delta G(x,\beta)}_{-\delta_\beta(x)} - G(x,\beta)\underbrace{\Delta G(x,\alpha)}_{-\delta_\alpha(x)}\, dx =$$

$$= \int_{\partial\Omega} \underbrace{G(x,\alpha)}_{=0}\partial_n G(x,\beta) - \underbrace{G(x,\beta)}_{=0}\partial_n G(x,\alpha)\, dS = 0$$

HUVUDSATSEN FÖR GREENFUNKTIONEN

Problemet $\begin{cases} -\Delta u = f & i\ \Omega. \\ u = g & på\ \partial\Omega \end{cases}$

$$u(x) = \int_\Omega G(x,\alpha) f(\alpha)\, d\alpha - \int_{\partial\Omega} \partial_n G(x,\alpha)\, g(\alpha)\, dS$$