Föreläsning 15

M.9 Lösning av differentialekvationer

Ex 15.1 Bestäm en kausal lösning till $y'+y = \begin{cases}1, & 0<t<1\\ 0, & \text{annars}\end{cases}$

Lösn $\mathcal{L}(y'+y) = \mathcal{L}(\text{HL}) = \int_0^1 e^{-st}\cdot 1\,dt = \left[\frac{e^{-st}}{-s}\right]_{t=0}^{1} = \frac{1}{s} - \frac{e^{-s}}{s}$

dvs $\mathcal{L}(y') + \mathcal{L}(y) = \frac{1-e^{-s}}{s}$

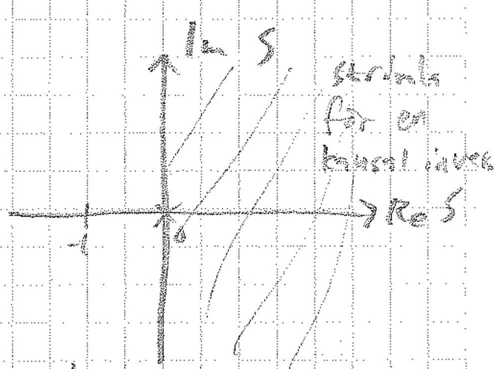
(28) $\Rightarrow s\mathcal{L}(y)+\mathcal{L}(y) = \frac{1-e^{-s}}{s}$

$\mathcal{L}(y) = \frac{1-e^{-s}}{s(1+s)}$

$\therefore y = \mathcal{L}^{-1}\left(\frac{1-e^{-s}}{s(1+s)}\right)$

Vilken definitionsstrimla ska vi välja?

$\frac{1-e^{-s}}{s(1+s)}$ har två enkla poler $0, -1$



Metod 1 Residyer

$$\operatorname*{Res}_{s=0}\left(\frac{e^{st}(1-e^{-s})}{s(1+s)}\right) = \left.\frac{e^{st}(1-e^{-s})}{[s(1+s)]'}\right|_{s=0} = 0$$

$$\operatorname*{Res}_{s=-1}\left(\frac{e^{st}(1-e^{-s})}{s(1+s)}\right) = \left.\frac{e^{st}(1-e^{-s})}{[s(1+s)]'}\right|_{s=-1} = \frac{e^{-t}(1-e)}{-1}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{1-e^{-s}}{s(s+1)}\right) = \left(0 + \frac{e^{-t}(1-e)}{-1}\right)\theta(t)$$

(33) $\theta(t) \leftrightarrow \frac{1}{s}$

$\theta(t-1) \leftrightarrow \frac{e^{-s}}{s}$

(37)

(26)

Metod 2

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{(s+1)-s}{s(s+1)} - e^{-s}\left(\frac{(s+1)-s}{s(s+1)}\right)\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+1} - e^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right)\right) = \theta(t) - e^{-t}\theta(t) - \left(\theta(t-1) - e^{-(t-1)}\theta(t-1)\right)$$

Ex 15.1 (6307) Typisk tenta fråga

Hitta den begränsade lösningen till

$$y''' + y'' - 2y = \delta'(t) \quad , \; -\infty < t < \infty$$

Lösn

Laplacetransformation ger

$$s^3 \mathcal{L}(y) + s^2 \mathcal{L}(y) - 2\mathcal{L}(y) = s\mathcal{L}(\delta) = s$$

$$\mathcal{L}(y) = \frac{s}{s^3+s^2-2}$$

$$y = \mathcal{L}^{-1}\left(\frac{s}{s^3+s^2-2}\right)$$

Vi ska välja strimla

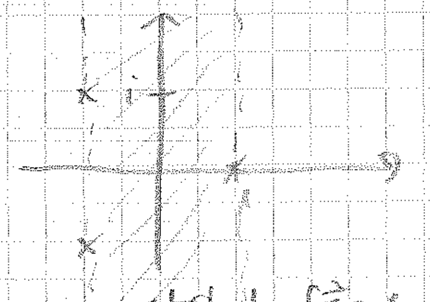
s^3+s^2-2 har ett nollställe $s_1 = 1$

$$\begin{array}{l} s^2+2s+2 \\ \hline s^3+s^2-2 \;\; | \; s-1 \\ s^3-s^2 \\ \hline 2s^2-2 \\ 2s^2-2s \\ \hline 2s-2 \\ 2s-2 \\ \hline 0 \end{array}$$

$\therefore \; s^3+s^2-2 = (s-1)(s^2+2s+2) =$

$= (s-1)((s+1)^2+1) = (s-1)(s+1-i)(s+1+i)$

Tre enkla poler: $1, \; -1 \pm i$



strimla för en begränsad lösning

$$\therefore \; y(t) = \mathcal{L}^{-1}\left(\frac{s}{s^3+s^2-2}\right) = \left.\begin{array}{l} \operatorname*{Res}_{s=-1+i}\left(\frac{e^{st}s}{s^3+s^2-2}\right) + \operatorname*{Res}_{s=-1-i}\left(\frac{e^{st}s}{s^3+s^2-2}\right) \end{array}\right\}\theta(t)$$

$$+ \operatorname*{Res}_{s=1}\left(\frac{e^{st}s}{s^3+s^2-2}\right)(\theta(t)-1)$$

$$\left.\frac{e^{st}s}{3s^2+2s}\right|_{s=1}$$

Metod 2 för beräkning av $y(t) = \mathcal{L}^{-1}\left(\frac{s}{(s-1)(s^2+2s+2)}\right)$

$$\frac{s}{(s-1)(s^2+2s+2)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+2}$$

$$s = A(s^2+2s+2) + (Bs+C)(s-1)$$

$s=1,\ s=0,\ s=-1 \Rightarrow A = \frac{1}{5},\ B = -\frac{1}{5},\ C = \frac{2}{5}$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{1}{5}\frac{1}{s-1} + \frac{-\frac{1}{5}s + \frac{2}{5}}{(s+1)^2+1^2}\right) = \frac{1}{5}e^{1\cdot t}\underbrace{(\theta(t)-1)}_{\text{ej } \theta(t)} + \mathcal{L}^{-1}\left(\frac{-\frac{1}{5}(s+1)+\frac{3}{5}}{(s+1)^2+1^2}\right)$$

$$= \frac{1}{5}e^{t}(\theta(t)-1) - \frac{1}{5}e^{-t}\cos t\,\theta(t) + \frac{3}{5}e^{-t}\sin t\,\theta(t)$$

<u>14.10</u> L.TI system

<u>Sats 15.1 (s 314)</u> $\mathcal{L}(f*g) = \mathcal{L}(f)\,\mathcal{L}(g)$

För LTI system $w(t) \xrightarrow{S} y(t)$ gäller $y(t) = (h*w)(t)$

$$\therefore \mathcal{L}(y) = \mathcal{L}(h*w) = \underbrace{\mathcal{L}(h)(s)}_{H(s)\text{ överföringsfunktion av } S}\mathcal{L}(w)$$

så att (a) $H(s) = \mathcal{L}(h)(s) = \dfrac{\mathcal{L}(y)}{\mathcal{L}(w)}$

(b) Då $w(t) = \theta(t)$ har vi

$\mathcal{L}(y) = \mathcal{L}(h)\cdot\mathcal{L}(\theta) = H(s)\frac{1}{s}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{H(s)}{s}\right)$ som kallas stegsvaret av systemet

<u>Ex 15.3</u> (s 312) S är LTI och kausalt så att

$$\sin t\cdot\theta(t) \xrightarrow{S} \sin 2t\cdot\theta(t)$$

Bestäm impulssvaret $h(t)$ och svaret på insignalen $w(t) = t\sin t\,\theta(t)$.

<u>Lösn</u>

$h(t) * (\sin t\,\theta(t)) = \sin 2t\cdot\theta(t)$

$\mathcal{L}(h)\cdot\mathcal{L}(\sin t\,\theta(t)) = \mathcal{L}(\sin 2t\,\underbrace{\theta(t)}_{=\theta(2t)})$

$\mathcal{L}(h)\cdot\frac{1}{s^2+1} = \frac{1}{|2|}\mathcal{L}(\sin t\,\theta(t))\left(\frac{s}{2}\right)$

$= \frac{1}{2}\frac{1}{\left(\frac{s}{2}\right)^2+1^2}$ $\quad\mathcal{L}(h) = 2\frac{s^2+1}{s^2+4} = 2\frac{(s^2+4)-3}{s^2+4} = 2 - 3\frac{2}{s^2+2^2}$

$h(t) = \mathcal{L}^{-1}\left(2 - 3\frac{2}{s^2+2^2}\right) = 2\delta(t) - 3\sin 2t\,\theta(t)$

$\therefore$ Svaret $y(t)$ på $w(t) = t\sin t\,\theta(t)$ uppfyller

$$y(t) = h * w = \left(2\delta - 3\sin 2t\,\theta\right) * \left(t\sin t\,\theta\right)$$

$$= 2t\sin t\,\theta(t) - 3\underbrace{(\sin 2t\,\theta) * (t\sin t\,\theta)}_{y_1(t)}$$

Där $\mathcal{L}(y_1) = \frac{2}{s^2+2^2}\mathcal{L}(t\sin t\,\theta(t))$

$$= \frac{2}{s^2+4}\left(-\frac{d}{ds}\mathcal{L}(\sin t\,\theta(t))(s)\right) = -\frac{2}{s^2+4}\cdot\frac{d}{ds}\left(\frac{1}{s^2+1^2}\right) =$$

$$\frac{2\cdot 2s}{(s^2+4)(s^2+1)^2}$$

$$\therefore\ y_1(t) = 4\,\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)(s^2+1)^2}\right)$$

Detta fungerar inte så bra

$\frac{s}{(s^2+4)(s^2+1)^2}$ har poler $\pm 2i$, $\pm i \leftarrow$ dubbelpoler