Föreläsning 13

kap 14

Laplacetransform

14.1–14.4

Def 13.1 (s 289)

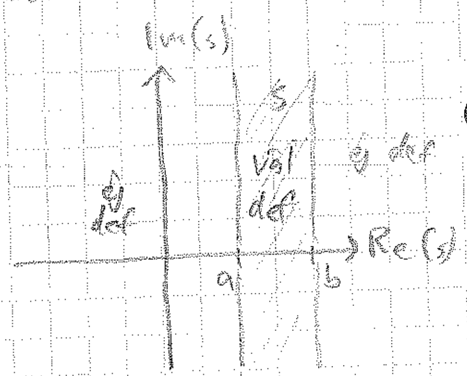
Laplacetransformen av $f(t)$ i $\mathbb{R}$ är

$$\mathcal{L}(f(t))(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{-st} f(t)\,dt$$

för alla komplexa tal s så att integralen är konvergent.

Anm: Definitionsmängden till $\mathcal{L}(f(t))(s)$ är en vertikal strimla $a < \mathrm{Re}(s) < b$ i komplexa planet.

Där a och b kan vara $\pm\infty$



Ex 13.1 Överföringsfunktionen till ett LTI system S är

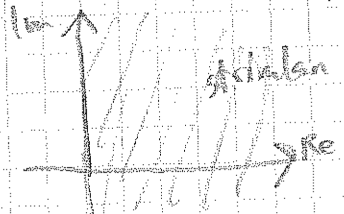
$$H(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{-st} h(t)\,dt \overset{\text{dvs}}{=} \mathcal{L}(h(t))(s)$$

Ex 13.2 (s 291)

a) $\mathcal{L}(\theta(t))(s) = \int_{-\infty}^{\infty} e^{-st}\theta(t)\,dt = \int_{0}^{\infty} e^{-st}\cdot 1\,dt \overset{s\neq 0}{=} \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty}$

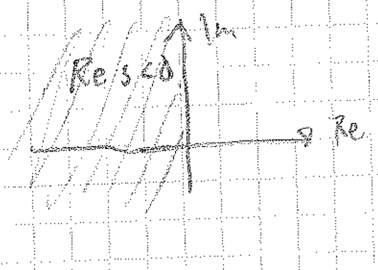
$\overset{\mathrm{Re}\, s>0}{=} \frac{0-1}{-s} = \frac{1}{s}$ för $\mathrm{Re}\, s > 0$

$|e^{-st}| = |e^{-(\mathrm{Re}\, s + i\,\mathrm{Im}\, s)t}| = e^{-\mathrm{Re}\, s\, t}$



b) $\mathcal{L}(\theta(t)-1)(s) = \int_{-\infty}^{\infty} e^{-st}(\theta(t)-1)\,dt = \int_{-\infty}^{0} e^{-st}(0-1)\,dt =$

$\left[\frac{e^{-st}}{s}\right]_{t=-\infty}^{0} \overset{\mathrm{Re}\, s<0}{=} \frac{1-0}{s} = \frac{1}{s}$ för $\mathrm{Re}\, s < 0$



Sats 13.1 (s 293 - s 296)

Om $f(t) \xrightarrow{\mathcal{L}} F(s)$ och $g(t) \xrightarrow{\mathcal{L}} G(s)$ så gäller

(24) $c_1 f(t) + c_2 g(t) \xrightarrow{\mathcal{L}} c_1 F(s) + c_2 G(s)$

(25) $f(at) \xrightarrow{\mathcal{L}} \frac{1}{|a|} F\left(\frac{s}{|a|}\right)$

(26) $f(t-a) \xrightarrow{\mathcal{L}} e^{-as} F(s)$, dvs $\mathcal{L}(f(t-a))(s) = e^{-as} \mathcal{L}(f(t))(s)$

(27) $e^{at} f(t) \xrightarrow{\mathcal{L}} F(s-a)$

(28) $f'(t) \xrightarrow{\mathcal{L}} s F(s)$

(29) $t f(t) \xrightarrow{\mathcal{L}} -\frac{d}{ds} F(s)$

(30) $f * g(t) \xrightarrow{\mathcal{L}} F(s) G(s)$

Ex 13.3 (s 293)

$\mathcal{L}(e^{at} \theta(t))(s) \overset{(27)}{=} \mathcal{L}(\theta(t))(s-a) \overset{\text{Ex 13.2}}{=} \frac{1}{s-a}$ där $\operatorname{Re}(s-a) > 0$
dvs $\operatorname{Re} s > \operatorname{Re} a$

Ex 13.4 (s 294)

$$\mathcal{L}(\sin(at)\,\theta(t))(s) = \mathcal{L}\left(\frac{e^{iat} - e^{-iat}}{2i}\theta(t)\right)(s)$$

$$= \frac{1}{2i}\mathcal{L}(e^{iat}\theta(t))(s) - \frac{1}{2i}\mathcal{L}(e^{-iat}\theta(t))(s)$$

$$\overset{(27)}{=} \frac{1}{2i}\mathcal{L}(\theta(t))(s-ia) - \frac{1}{2i}\mathcal{L}(\theta(t))(s+ia)$$

$$\overset{\substack{\operatorname{Re}(s) = \operatorname{Re}(s-ia) > 0 \\ \operatorname{Re}(s) = \operatorname{Re}(s+ia) > 0}}{=} \frac{1}{2i}\frac{1}{s-ia} - \frac{1}{2i}\frac{1}{s+ia} = \frac{1}{2i}\frac{(\cancel{s}+ia) - (\cancel{s}-ia)}{(s-ia)(s+ia)}$$

$$= \frac{1}{2i}\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2} \quad \text{för } \operatorname{Re} s > 0$$

$\sigma = \operatorname{Re} s$ i formelblad

Alla exempel står i formelbladet

Ex 13.5 $\mathcal{L}(t \sin at\, \theta(t))(s) \overset{(29)}{=} -\frac{d}{ds}\mathcal{L}(\sin at\, \theta(t))(s) \underset{\mathrm{Re}\, s>0}{\overset{\text{Ex } 13.4}{=}} -\frac{d}{ds}\frac{a}{s^2+a^2}$

$= \frac{-a(-2s)}{(s^2+a^2)^2} = \frac{2as}{(s^2+a^2)^2}$ för $\mathrm{Re}\, s > 0$

14.7 Laplacetransformen av en tempererad distribution u är definierad genom $\mathcal{L}(u)(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{-st}\, u(t)\, dt$

Ex 13.6 (s 300)

a) $\mathcal{L}(\delta(t))(s) = \int_{-\infty}^{\infty} e^{-st}\delta(t)\,dt = e^{-s\cdot 0} = 1$ dvs $\mathcal{L}(\delta) = 1$

b) $\mathcal{L}(\delta'(t))(s) \overset{(28)}{=} s\mathcal{L}(\delta(t))(s) = s\cdot 1 = s$

$\therefore \mathcal{L}(\delta^{(k)})(s) = s^k$ för $k = 0, 1, 2 \ldots$

och $c_n\delta^{(n)}(t) + c_{n-1}\delta^{(n-1)}(t) + \ldots + c_1\delta'(t) + c_0\delta(t)$

$\xrightarrow{\mathcal{L}} c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0 s^0$

14.5 Inversionsformeln

Def (s 298) Om $f(t) \xrightarrow{\mathcal{L}} F(s)$ så kallas $f(t)$ för den inversa Laplacetransformen av $F(s)$. Vi skriver $\mathcal{L}^{-1}(F(s))(t) = f(t)$

dvs $f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)$

$f(t) \xrightarrow{\mathcal{F}} F(\omega)$

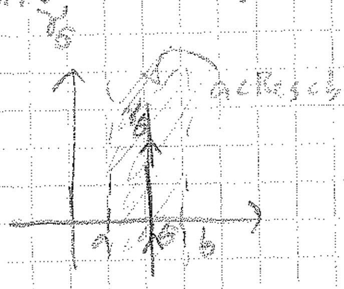
$\mathcal{F}(f(t))(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt$

$\mathcal{F}^{-1}(F(\omega))(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega$

Sats 3.2 (s 297)

$f(t) = \mathcal{L}^{-1}(F(s))(t) = \frac{1}{2\pi i}\int_{\gamma_\sigma} e^{st} F(s)\, ds$

Där kurvan γ_σ är en vertikal linje som ligger i definitionsstrimlan $a < \mathrm{Re}\, s < b$ av $F(s)$



Ex 13.6 Bestäm $\mathcal{L}^{-1}(F(s))(t)$ för

(a) $F(s) = \frac{1}{s}$, $\operatorname{Re} s > 0 \overset{\text{Ex 13.2}}{\Longrightarrow} \mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) = \theta(t)$

$$\theta(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \frac{1}{s}, \quad \operatorname{Re} s > 0$$

(b) $F(s) = \frac{1}{s}$, $\operatorname{Re} s < 0 \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) = \theta(t) - 1$

(c) $F(s) = \frac{1}{s^2}$, $\operatorname{Re} s > 0 \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = \mathcal{L}^{-1}\left(-\frac{d}{ds}\frac{1}{s}\right)(t)$

$$\overset{(29)}{=} t\,\mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) \overset{\operatorname{Re} s > 0}{=} t\,\theta(t)$$

$$\boxed{(29)\quad \mathcal{L}^{-1}\left(-\frac{d}{ds}F(s)\right)(t) = t\,\mathcal{L}^{-1}(F(s))(t)}$$

(d) $\underbrace{\mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right)(t)}_{F(s) = \frac{1}{(s-2)^2},\ \operatorname{Re} s > 2 \Rightarrow} = \mathcal{L}^{-1}\left(-\frac{d}{ds}\frac{1}{s-2}\right)(t) \overset{(29)}{=} t\,\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)(t)$

$$\overset{(27)}{=} t\,e^{2t}\,\mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) = t\,e^{2t}\,\theta(t)$$

Ex 13.7 Bestäm den kausala inversen Laplacetransformen till

(a) $\frac{s^2}{s+1}$ (b) $\frac{1}{(s+1)(s+2)}$

Lösn (a)

$$\begin{array}{l|l} s^2 & s+1 \\ \hline s^2+s & s-1 \leftarrow \text{kvot} \\ -s & \\ -s-1 & \\ 1 \leftarrow \text{rest} & \end{array}$$

$$\therefore \frac{s^2}{s+1} = \overset{\text{kvot}}{\boxed{s-1}} + \frac{1^{\leftarrow \text{rest}}}{s+1}$$

$$\mathcal{L}^{-1}\left(\frac{s^2}{s+1}\right)(t) = \mathcal{L}^{-1}\left(s-1+\frac{1}{s+1}\right)(t) \overset{\text{kausal}}{=} \delta'(t) - \delta(t) + e^{-t}\,\theta(t)$$