22/2-2012. Rep. Speciella funktioner

(12) För Re $z > 0$ definieras gammafunktionen Γ av $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt = \int_0^\infty e^{(z-1)\ln t} e^{-t}\,dt$.

Partiell integration ger

$$\Gamma(z+1) = z\cdot\Gamma(z)$$

Upprepn. $\Gamma(n+1) = n!$ då n pos. heltal.

Analytisk i HHP

Antag Re $z < 0$ och $z \neq 0, -1, -2, -3, \dots$

Välj n så stort att Re$(z+n) > 0$

$\Gamma(z+n) = (z+n-1)\Gamma(z+n-1) = (z+n-1)(z+n-2)\Gamma(z+n-2) =$

$= (z+n-1)(z+n-2)\dots z\Gamma(z)$

$\Rightarrow \Gamma(z) = \frac{\Gamma(z+n)}{(z+n-1)(z+n-2)\dots z}$

Låt m vara ett annat positivt heltal med Re$(z+m) > 0$

D.s.s.

$$\Gamma(z) = \frac{\Gamma(z+m)}{(z+m-1)(z+m-2)\dots z}$$

$$t^k\theta(t) \xrightarrow{\mathcal{L}} \frac{k!}{s^{k+1}}$$

Sätt $k_\alpha(t) = t^{\alpha-1}\theta(t)$, Re $\alpha > 0$

$\mathcal{L}(k_\alpha) = ?$

Säg att $s > 0$

I) $\mathcal{L}(k_\alpha(s)) = \int_0^\infty t^{\alpha-1} e^{-st}\,dt = \left[\begin{matrix} st = x \\ dt = \frac{dx}{s}\end{matrix}\right] = \int_0^\infty s^{1-\alpha} x^{\alpha-1} e^{-x}\frac{dx}{s} =$

$= s^{-\alpha}\int_0^\infty x^{\alpha-1}e^{-x}\,dx = \frac{\Gamma(\alpha)}{s^\alpha}$

II) Säg nu att s är komplext med Re $s > 0$

Då gäller $\mathcal{L}(k_\alpha(s)) = \frac{\Gamma(\alpha)}{s^\alpha}$

Betafunktionen

$$k_\alpha * k_\beta \xrightarrow{\mathcal{L}} \frac{\Gamma(\alpha)}{s^\alpha}\cdot\frac{\Gamma(\beta)}{s^\beta} = \underbrace{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}_{=B(\alpha,\beta)}\cdot\frac{\Gamma(\alpha+\beta)}{s^{\alpha+\beta}}$$

~~Alls~~ Inverstransformering ger

$k_\alpha * k_\beta = B(\alpha,\beta)k_{\alpha+\beta}$ (4)

Liten formelsamling

$\Gamma(\frac12) = \sqrt{\pi}$, (5)

$B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1}t^{\beta-1}\,dt$, Re $\alpha > 0$, Re $\beta > 0$ (6)

$\Gamma(\alpha)\cdot\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha}$, $\alpha \neq 0, \pm1, \pm2, \dots$ (7)

$\Gamma(\frac12) = \int_0^\infty t^{\frac12-1}e^{-t}\,dt = \int_0^\infty \frac{e^{-t}}{\sqrt t}\,dt = \left[\begin{matrix}t = x^2\\ dt = 2x\,dx\end{matrix}\right] =$

$= \int_0^\infty \frac{e^{-x^2}}{x}\cdot 2x\,dx = 2\int_0^\infty e^{-x^2}\,dx = \sqrt\pi$

$\Gamma(\frac32) = \Gamma(\frac12+1) = \frac12\cdot\Gamma(\frac12) = \frac12\sqrt\pi$

Bevis av (6) (4) kan skrivas $\int_{-\infty}^\infty \underbrace{k_\alpha(t-\tau)}_{\theta(t-\tau)}\underbrace{k_\beta(\tau)}_{\tau^{\beta-1}\theta(\tau)}\,d\tau =$

$= B(\alpha,\beta)k_{\alpha+\beta}(t)$

$\int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1}\,d\tau = B(\alpha,\beta)t^{\alpha+\beta}$ om $t > 0$

I) $t = 1$ ger (6).

Ex) Lös $\int_0^x \sqrt{x-y}\,u(y)\,dy = x$ där $x > 0$

Lösning: $\theta(x)\int_0^x (x-y)^{3/2-1}u(y)\,dy = x\,\theta(x)$

$\int_{-\infty}^\infty \underbrace{(x-y)^{3/2-1}\theta(x-y)}_{k_{3/2}(x-y)}\cdot\theta(y)u(y)\,dy = k_{3/2} * (\theta u) = x\,\theta(x)$

Laplace ger $\frac{\Gamma(\frac32)}{s^{3/2}}\cdot U = \frac{1}{s^2}$ där $U = \mathcal{L}(\theta u)$

$U = \frac{1}{s^{1/2}}\frac{1}{\Gamma(\frac32)} = \frac{\Gamma(\frac12)}{s^{1/2}}\cdot\frac{1}{\Gamma(\frac12)\cdot\Gamma(\frac32)} = \frac{\Gamma(\frac12)}{s^{1/2}}\cdot\frac{1}{\sqrt\pi\,\frac12\sqrt\pi}$

$\xrightarrow{\mathcal{L}^{-1}} k_{\frac12}(x)\cdot\frac{2}{\pi} = x^{\frac12-1}\theta(x)\cdot\frac{2}{\pi}$

$u(x) = \frac{1}{\sqrt x}\cdot\frac{2}{\pi}$

Besselfunktioner

För fixt r är $u(r,\theta) = e^{ir\sin\theta}$ periodisk med $T = 2\pi$

$e^{ir\sin\theta} = u(r,\theta) = \sum_{n=-\infty}^\infty c_n(r)e^{in\theta}$, $c_n(r) \to J_n(r)$

$J_n(r) = c_n(r) = \frac{1}{2\pi}\int_{-\pi}^\pi e^{in\theta}e^{ir\sin\theta}\,d\theta$

$e^{ir\sin\theta} = \sum_{n=-\infty}^\infty J_n(r)e^{in\theta}$

Anm: $r = 0$ ger $1 = \sum_{n=-\infty}^\infty J_n(0)e^{in\theta} \Rightarrow J_0(0) = 1$, $J_n(0) = 0$ om $n \neq 0$.

Med $u = e^{iy}$ blir $-\Delta u = e^{iy} = u$

Med polära koordinater blir $u = e^{ir\sin\theta}$

$\frac{\partial^2u}{\partial r^2} + \frac1r\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2} + u = 0$

$\sum\underbrace{\left(J_n''(r) + \frac1r J_n'(r) - \frac{n^2}{r^2}J_n(r) + J_n(r)\right)}_{=0}e^{in\theta} = 0$

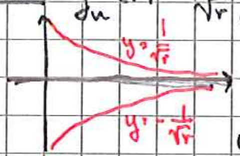
Alltså löser J_n diff. ekv.

$v''(r) + \frac1r v'(r) + \left(1 - \frac{n^2}{r^2}\right)v(r) = 0$ (11)

$\operatorname{Im} J_n(r) = 0 \Leftrightarrow J_n(r) = \frac{1}{2\pi}\int_{-\pi}^\pi \left((\cos n\theta - i\sin n\theta)(\cos(r\sin\theta) + i\sin(r\sin\theta))\right)d\theta$

$J_{-n}(r) = (-1)^n J_n(r)$

Anm. $J_n(r) \approx \frac{1}{\sqrt r}(a\cos r + b\sin r)$ om r stort.



Troliggörande: Sätt $y(r) = \sqrt r\cdot J_n(r)$

$y'(r) = \frac{1}{2\sqrt r}J_n(r) + \sqrt r J_n'(r)$

$y''(r) = -\frac{1}{4r^{3/2}}J_n(r) + \frac{1}{\sqrt r}J_n'(r) + \sqrt r J_n''(r)$

$y''(r) + \left(1 - \frac{n^2-\frac14}{r^2}\right)y(r) = \sqrt r\left(\underbrace{J_n''(r) + \frac1r J_n'(r)}_{(\frac{n^2}{r^2}-1)J_n(r)} - \frac{1}{4r^2}J_n(r)\right) + \left(1 - \frac{n^2-\frac14}{r^2}\right)\sqrt r J_n(r)$

Om r stort.

$y''(r) + y(r) \approx 0$

$y(r) \approx a\cos r + b\sin r$

Nollställen till $J_0(r)$

2,4 ; 5,5 ; 8,6 ; 11,8.

$J_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} e^{ir\sin\theta} \, d\theta$. Då $n \geq 0$ och heltal är $J_n(r)$ en hel analytisk funktion

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

$$a_k = \frac{f^{(k)}(a)}{k!}$$

$$J_n(r) = \left(\frac{r}{2}\right)^n \sum \frac{1}{k!(n+k)!} \left(-\frac{r^2}{4}\right)^k$$

$$J_n(r) = \sum \frac{J_n^{(m)}(0)}{m!} r^m$$

$$J_n^{(m)}(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} (i\sin\theta)^m e^{ir\sin\theta} \, d\theta$$

$$J_n^{(m)}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} i^m \sin^m\theta \, d\theta = \frac{i^m}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^m d\theta$$

$$= \frac{1}{2^m 2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \sum_{k=0}^{m} \binom{m}{k} e^{i(m-k)\theta} (-1)^k e^{-ik\theta} \, d\theta =$$

$$\left((a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k \right)$$

$$= \frac{1}{2^m 2\pi} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \underbrace{\int_{-\pi}^{\pi} e^{i(m-2k-n)\theta} \, d\theta}_{=0 \text{ utom då } m-2k-n=0}$$

Def. för $\nu \neq$ positivt heltal. Då $\nu \neq$ neg. heltal sätter vi

$$J_\nu(r) = \left(\frac{r}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(-\frac{r^2}{4}\right)^k$$

$$v''(r) + \frac{1}{r} v'(r) + \left(1 - \frac{\nu^2}{r^2}\right) v(r) = 0. \quad (*)$$

Om $\nu \neq$ heltal så är en allmän lösning till $*$ av formen $aJ_\nu(r) + bJ_{-\nu}(r)$

$$J_{-n}(r) = (-1)^n J_n(r).$$

Sammanfattning. Sats S1, S2