Föreläsning 11

Ex 11.1 $\widehat{e^{-3t^2}\cos 2t}(w) = \widehat{e^{-3t^2}\cdot\frac{e^{i2t}+e^{-i2t}}{2}}(w) = \frac{1}{2}\left(\widehat{e^{-3t^2}\cdot e^{i2t}}(w) + \widehat{e^{-3t^2}e^{-i2t}}(w)\right)$

(10) formelblad $\frac{1}{2}\widehat{e^{-3t^2}}(w-2) + \frac{1}{2}\widehat{e^{-3t^2}}(w+2) =$

$\frac{1}{2}\widehat{e^{-(\sqrt{3}t)^2}}(w-2) + \frac{1}{2}\widehat{e^{-(\sqrt{3}t)^2}}(w+2) \overset{(8)}{=} \frac{1}{2}\frac{1}{\sqrt{3}}\widehat{e^{-t^2}}\left(\frac{w-2}{\sqrt{3}}\right)$ ← Obs, $\frac{w}{\sqrt{3}}-2$

$+\frac{1}{2}\frac{1}{\sqrt{3}}\widehat{e^{-t^2}}\left(\frac{w+2}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}}\sqrt{\pi}\, e^{-\left(\frac{w-2}{\sqrt{3}}\right)^2\frac{1}{4}} + \frac{1}{2\sqrt{3}}\sqrt{\pi}\, e^{-\left(\frac{w+2}{\sqrt{3}}\right)^2\frac{1}{4}}$

$= \frac{1}{2}\sqrt{\frac{\pi}{3}}\left(e^{-\frac{(w-2)^2}{12}} + e^{-\frac{(w+2)^2}{12}}\right)$

13.5 Invers Fouriertransformation

$$f(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} F(w)$$

Def 11.1 (s 263)

Om $F = \mathcal{F}(f)$, så kallas f inversen Fouriertransformationen till F. Vi skriver $\mathcal{F}^{-1}(F) = f$

$\mathcal{F}(f(t))(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t)\,dt = \hat{f}(w)$

Sats 11.1 (Fouriers inversionsformel s 262 och 264)

$f, \hat{f} \in L_1 \Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwt}\hat{f}(w)\,dw \overset{\text{dvs}}{=} \mathcal{F}^{-1}(\hat{f}(w))(t)$

för alla t, där f är kontinuerlig

Två viktiga likheter:

Följdsats ① $\mathcal{F}^{-1}(F(w))(t) = \frac{1}{2\pi}\mathcal{F}(F(w))(-t)$

② $\widehat{\widehat{f(t)}} = 2\pi f(-t)$

Dvs $\frac{1}{2\pi}F(w) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\leftarrow}} f(t) \xrightarrow{\mathcal{F}} F(w) \xrightarrow{\mathcal{F}} 2\pi f(-t)$

Bevis ① $\mathcal{F}^{-1}(F(w))(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwt}F(w)\,dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iw(-t)}F(w)\,dw$

$= \frac{1}{2\pi}\mathcal{F}(F(w))(-t)$

Bevis (4) $\widehat{\widehat{f(t)}} = \int_{-\infty}^{\infty} e^{-i\omega t}\widehat{f(t)}(\omega)\,d\omega = 2\pi \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(-t)}\widehat{f(t)}(\omega)\,d\omega$

$= 2\pi f(-t)$

Ex 11.2 $e^{-|t|} \xrightarrow[\mathcal{F}]{\text{(13) ger}} \frac{2}{1+\omega^2} \xrightarrow{\mathcal{F}} 2\pi e^{-|-t|}$

$\therefore \frac{2}{1+\omega^2} \xrightarrow{\mathcal{F}} 2\pi e^{-|t|}$

$\frac{1}{1+\omega^2} \xrightarrow{\mathcal{F}} \pi e^{-|t|}$

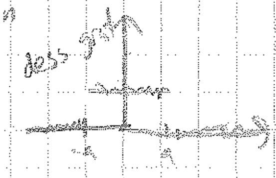
Ans $\frac{1}{1+t^2} \xrightarrow{\mathcal{F}} \pi e^{-|\omega|}$

Ex 11.3 (20) ger

$\theta(t+a) - \theta(t-a) \xrightarrow{\mathcal{F}} 2\frac{\sin a\omega}{\omega} \xrightarrow{\mathcal{F}} 2\pi(\theta(-t+a) - \theta(-t-a))$

$= 2\pi(\theta(t+a) - \theta(t-a))$

$\theta(t+a) - \theta(t-a)$ är jämn



$\therefore \frac{\sin a\omega}{\omega} \xrightarrow{\mathcal{F}}$

$\pi(\theta(t+a) - \theta(t-a))$ Ans $\frac{\sin at}{t} \xrightarrow{\mathcal{F}} \pi(\theta(\omega+a) - \theta(\omega-a))$

$\frac{1}{2\pi}F(-\omega) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} f(t) \xrightarrow{\mathcal{F}} F(\omega) \xrightarrow{\mathcal{F}} 2\pi f(-t)$

Ex 11.4 Beräkna inversen Fouriertransformen av $F(\omega) = e^{-\frac{\omega^2}{2}}$

Lösn (byte $\omega \to t$) $\widehat{e^{-\frac{t^2}{2}}}(\omega) = \widehat{e^{-\left(\frac{1}{\sqrt{2}}t\right)^2}}(\omega) = \frac{1}{\frac{1}{\sqrt{2}}}\widehat{e^{-t^2}}\left(\frac{\omega}{\frac{1}{\sqrt{2}}}\right)$

$\overset{(19)}{=} \sqrt{2}\sqrt{\pi}\, e^{-\left(\frac{\omega}{\frac{1}{\sqrt{2}}}\right)^2 \cdot \frac{1}{4}} = \sqrt{2\pi}\, e^{-\frac{\omega^2}{2}}$

Dvs $\mathcal{F}\left(e^{-\frac{t^2}{2}}\right)(\omega) = \sqrt{2\pi}\, e^{-\frac{\omega^2}{2}}$

så är $\mathcal{F}^{-1}\left(e^{-\frac{\omega^2}{2}}\right)(t) = \frac{1}{2\pi}\mathcal{F}\left(e^{-\frac{\omega^2}{2}}\right)(-t) = \frac{1}{2\pi}\sqrt{2\pi}\, e^{-\frac{(-t)^2}{2}}$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

Ex 11.6 Hitta en lösning till $y'+y=e^{-2t}\theta(t)$ $\qquad \theta(t)=\theta(2t)$

Lösn $\widehat{y'(t)+y(t)}=\widehat{e^{-2t}\theta(t)}$

$i\omega\hat{y}(\omega)+\hat{y}(\omega)=\widehat{e^{-2t}\theta(2t)}(\omega) \;\Big|\; (i\omega+1)\hat{y}(\omega)=\frac{1}{2}\widehat{e^{-t}\theta(t)}\left(\frac{\omega}{2}\right) \;\Big|\; (1+i\omega)\hat{y}(\omega)=$

$\overset{(\text{ii})}{=}\frac{1}{2}\frac{1}{1+i\frac{\omega}{2}}=\frac{1}{2+i\omega}$

$\hat{y}(\omega)=\frac{1}{(1+i\omega)(2+i\omega)}=\frac{(2+i\omega)-(1+i\omega)}{(1+i\omega)(2+i\omega)}$

$\hat{y}(\omega)=\frac{1}{1+i\omega}-\frac{1}{2+i\omega}$

$y(t)=\mathcal{F}^{-1}(\hat{y}(\omega))(t)=\mathcal{F}^{-1}\left(\frac{1}{1+i\omega}-\frac{1}{2+i\omega}\right)(t)\overset{(\text{T6})}{=}e^{-t}\theta(t)-e^{-2t}\theta(t)$

13.7

Sats 11.7 (Faltningssatsen s 267)

$f,g\in L_1 \Rightarrow \widehat{f*g}=\hat{f}\cdot\hat{g}$

Ex 11.5 (s 270)

Beräkna $g(t)=e^{-t^2}*e^{-t^2}$ m.h.a. Fouriertransform

Lösn $\hat{g}=\widehat{e^{-t^2}*e^{-t^2}}=\widehat{e^{-t^2}}\cdot\widehat{e^{-t^2}}\overset{(\text{T9})}{=}\sqrt{\pi}\,e^{-\frac{\omega^2}{4}}\sqrt{\pi}\,e^{-\frac{\omega^2}{4}}=\pi e^{-\frac{\omega^2}{2}}$

$\therefore g(t)=\mathcal{F}^{-1}\left(\pi e^{-\frac{\omega^2}{2}}\right)(t)=\frac{1}{2\pi}\mathcal{F}\left(\pi e^{-\frac{\omega^2}{2}}\right)(-t)$

$=\frac{1}{2}\mathcal{F}\left(e^{-\left(\frac{1}{\sqrt{2}}\omega\right)^2}\right)(t)=\frac{1}{2}\frac{1}{\frac{1}{\sqrt{2}}}\mathcal{F}\left(e^{-\omega^2}\right)\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)$

$\overset{(\text{T9})}{=}\frac{1}{\sqrt{2}}\sqrt{\pi}\,e^{-\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)^2\frac{1}{4}}=\sqrt{\frac{\pi}{2}}\,e^{-\frac{t^2}{2}}$

13.8 Sats 11.3 (Parsevals formel) (s 273)

$f,g\in L_2=\{h(t):\int_{-\infty}^{\infty}|h(t)|^2\,dt<\infty\}$

$$\int_{-\infty}^{\infty}\overline{f(t)}\,g(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\hat{f}(\omega)}\,\hat{g}(\omega)\,d\omega$$

Speciellt gäller $\int_{-\infty}^{\infty}|f(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega$

Ex 11.6 (s 272)

Beräkna $\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt$

Lösn (20) ger $\int_{-\infty}^{\infty} \left(\theta(t+1) - \theta(t-1)\right)^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2 \sin w}{w}\right)^2 dw$

$\therefore \frac{4}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2 w}{w^2}\, dw = \int_{-1}^{1} 1^2\, dt$

$\Longrightarrow \int_{-\infty}^{\infty} \frac{\sin^2 w}{w^2}\, dw = \pi$

dvs $\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \pi$