Föreläsning 4 Lv 2

$$\int f(g(x))\,g'(x)\,dx \overset{t=g(x)}{=} \int f(t)\,dt$$

Ex. $\displaystyle\int x\sqrt{x+1}\,dx \underset{dt=dx}{\overset{t=x+1}{=}} \int (t-1)\,t^{\frac{1}{2}}\,dt = \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}} + C = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

Alt. $\displaystyle\int x\sqrt{x+1}\,dx \overset{s=\sqrt{x+1}}{=} \int (s^2-1)s\cdot 2s\,ds = 2\int (s^4-s^2)\,ds$

$x = s^2-1$
$dx = 2s\,ds$

$$= 2\left(\frac{s^5}{5} - \frac{s^3}{3}\right) + C = 2\left(\frac{(x+1)^5}{5} - \frac{(x+1)^3}{3}\right) + C$$

Sats. Formeln för partialintegration

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$\int f(x)g(x)\,dx = \int F'(x)\,g(x)\,dx = F(x)g(x) - \int F(x)\,g'(x)\,dx$$

Bevis av 1 $\displaystyle\int \big(f(x)g(x)\big)'\,dx = f(x)g(x) + C$

Men $\displaystyle VL = \int \big(f'(x)g(x) + f(x)g'(x)\big)\,dx = \int f'(x)\,g(x)\,dx + \int f(x)g'(x)\,dx = HL$

som ger $\displaystyle\int f(x)\,g'(x)\,dx = f(x)g(x) + C - \int f'(x)\,g(x)\,dx$

Ex.
a) $\displaystyle\int xe^x\,dx = \int x(e^x)'\,dx = xe^x - \int (x)'e^x\,dx = xe^x - e^x$

b) $\displaystyle\int x^2e^x\,dx = \int x^2(e^x)'\,dx = x^2\cdot e^x - \int 2xe^x\,dx = x^2e^x - \int 2x\,(e^x)'\,dx$

$$= x^2e^x - \left(2xe^x - \int 2e^x\,dx\right) = x^2e^x - 2xe^x + 2e^x + C$$

Ex. 12,12

$$\int \ln x\,dx = \int (x)'\cdot\ln x\,dx = x\cdot\ln x - \int x\cdot(\ln x)'\,dx = x\cdot\ln x - \int dx$$

$$= x\cdot\ln x - x + C$$

Ex. 12.14

Beräkna $I = \int e^{2x} \cos x \, dx =$

$\boxed{\text{Lösning:}}$ $I = \int e^{2x} (\sin x)' dx = e^{x} \cdot \sin x - \int \underbrace{e^{2x} \cdot 2}_{(e^{2x})'} \cdot \sin x \, dx$

$= e^{2x} \cdot \sin x - 2\int e^{2x} (-\cos x)' dx = e^{2x} \cdot \sin x - 2\left(e^{2x}(-\cos x) - \int e^{2x} \cdot 2 (-\cos x) dx\right)$

$= e^{2x} \cdot \sin x + 2e^{2x} \cdot \cos x - 4\underbrace{\int e^{2x} \cos x \, dx}_{I}$

$I = e^{2x} \cdot \sin x + 2e^{2x} \cdot \cos x - 4I$ Lös ut I!

Ex. 4. $\int \frac{x}{x^4+1} dx \overset{t=x^2}{=} \frac{1}{2} \int \underbrace{\frac{1}{t^2+1}}_{(\arctan t)'} dt = \frac{1}{2} \arctan(t) + C$

$dt = 2x dx$

$= \frac{1}{2} \arctan(x^2) + C$

Problem: Hur beräknar man $\int \frac{p(x)}{q(x)} dx$? (polynom, polynom)

$\boxed{\frac{p}{q} = \text{kvot} + \frac{\text{rest}}{q}}$

Lösningsmetod: (Partialbråksuppdelning)

Steg 1. Genom polynomdivision kan vi anta att täljarens grad < nämnarens grad

Steg 2. Faktorisera nämnaren $q(x)$ som produkt av polynom med grad 1 lr. 2.

Steg 3. Varje första grads faktor $(ax+b)^n$ ger $\boxed{\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_n}{(ax+b)^n}}$

Varje "andragrads" faktor $(ax^2+bx+c)^n$ ger

$$\boxed{\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}}$$

Steg 4. Bestämma alla dessa konstanter $A_1 \dots B_1 \dots$

Steg 5. Integrera varje term.

Ex. Beräkna $I = \int \frac{x^4+3x+1}{x^4+x^2} dx$

Lösning: Steg 1. Polynomdivision $\frac{x^4+3x+1}{x^4+x^2}$

$I = \int \left(1 + \frac{-x^2+3x+1}{x^4+x^2}\right) dx = x + \int \frac{-x^2+3x+1}{x^4+x^2} dx$

Steg 2 och 3.

$$\frac{-x^2+3x+1}{x^4+x^2} = \frac{-x^2+3x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

Steg 4. $\because$ $-x^2+3x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$

dvs. $-x^2+3x+1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$

$$= (A+C)x^3 + (B+D)x^2 + Ax + B$$

som ger $\begin{cases} A+C=0 \\ B+D=-1 \\ A=3 \\ B=1 \end{cases}$ Ur detta kan vi enkelt bestämma A, B, C, D.

$A=3 \quad B=1 \quad C=-3 \quad D=-2$

$$\therefore I = x + \int\left(\frac{3}{x} + \frac{1}{x^2} + \frac{-3x-2}{x^2+1}\right)dx = x + 3\ln|x| + \frac{x^{-2+1}}{-2+1} - 3\overbrace{\int\frac{x}{x^2+1}dx}^{*}$$

$$-2\int\frac{1}{x^2+1}dx \Big\} \arctan x + C$$

$*$ $$\int\frac{x}{x^2+1}dx \underset{dt=2x\,dx}{\overset{t=x^2+1}{=}} \frac{1}{2}\int\frac{1}{t}dt = \frac{1}{2}\ln|t| + C_1$$

$$= \frac{1}{2}\ln(x^2+1) + C_1 \quad *$$

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12,4 Fortsättning

Ex. $\displaystyle\int\frac{x^2+1}{x(x+1)^2}dx$

Lösning: (Täljarens grad < nämnarens grad och faktorisering av nämnare) Ok!

Sätt $\dfrac{x^2+1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$ $\quad\because$ $x^2+1 = A(x+1)^2 + Bx(x+1) + Cx$

För $x=0$ gäller $1 = A+0+0 \iff A=1$

För $x=-1$ gäller $2 = 0+0-C \iff C=-2$

Nu kan man stoppa in vilket tal som helst för att få ut värdet av B.

För $x=1$ gäller $2 = 4A+2B+C \Rightarrow B=0$

$$\because \int\frac{x^2+1}{x(x+1)}dx = \int\frac{1}{x} + \frac{-2}{(x+1)^2}dx = \ln|x| - 2\int\frac{1}{(x+1)^2}dx$$

$$= \ln|x| + 2\cdot\frac{1}{x+1} + C$$

basbyte $x+1=t$

Ex. Tentauppgift 0,6p

stoppar in i slutet

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx \overset{t=\sqrt{x}}{=} \int \frac{t}{1+t}\,2t\,dt = \boxed{2} \int \frac{t^2}{1+t}\,dt$$

$x=t^2$
$dx = 2t\,dt$

$$* \quad \frac{t^2}{t+1} = t-1+\frac{1}{t+1} \quad *$$

ny integral $\int \left(t-1+\frac{1}{t+1}\right) dt$

$$= \frac{t^2}{2} + t + \ln|t+1| + C$$

stoppar in 2

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = t^2 - 2t + 2\ln|t+1| + C_1$$

C_1 då $C_1 = 2C$

$$= x - 2\sqrt{x} + 2\ln|\sqrt{x}+1| + C_1$$

$(\sqrt{x})^2 = x \qquad \sqrt{x^2} = |x|$

12,5 Sats

① $\int f(\cos x)\sin x\,dx = \begin{bmatrix} t=\cos x \\ dt=-\sin x\,dx \end{bmatrix} - \int f(t)\,dt$

② $\int f(\sin x)\cos x\,dx = \begin{bmatrix} t=\sin x \\ dt=\cos x\,dx \end{bmatrix} \int f(t)\,dt$

Ex. $\int \frac{\sin x}{2\cos x+1}\,dx = \begin{bmatrix} t=\cos x \\ dt=-\sin x\,dx \end{bmatrix} - \int \frac{dt}{2t+1} = -\frac{\ln|2t+1|}{2} + C$

Alternativ lösning: $\int \frac{\sin x}{2\cos x+1}\,dx = \begin{bmatrix} t=2\cos x+1 \\ dt=2(-\sin x)dx \end{bmatrix} = -\frac{1}{2}\int \frac{dt}{t}$

$$= -\frac{1}{2}\ln|t| + C = -\frac{1}{2}\ln|2\cos x+1| + C$$

Ex. 12,23

udda tal

$$\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx = \begin{bmatrix} t=\sin x \\ dt=\cos x\,dx \end{bmatrix}$$

$$= \int (1+t^2)^2\,dt = \int (1-2t^2+t^4)\,dt = t - 2\frac{t^3}{3} + \frac{t^5}{5} + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Ex. 12,22

jämn

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}x - \frac{1}{2}\sin 2x + C$$

Ex. $\int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{1-\cos^2 x} dx = \begin{bmatrix} t = \cos x \\ dt = -\sin x\, dx \end{bmatrix}$

$= -\int \frac{dt}{1-t^2} = \int \frac{dt}{(t-1)(t+1)} = \frac{1}{2}\int \frac{(t+1)-(t-1)}{(t-1)(t+1)} dt$

$= \frac{1}{2}\int\left(\frac{1}{t-1} - \frac{1}{t+2}\right)dt = \frac{1}{2}(\ln|t-1| - \ln|t+1|) + C$

$= \frac{1}{2}(\ln|\cos x - 1| - \ln|\cos x + 1| + C$

Metoden fungerar också för $\int \frac{1}{\cos x} dx$

Alternativ lösning

$\int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \begin{bmatrix} t = \frac{x}{2} \\ dt = \frac{1}{2}dx \end{bmatrix} = \int \frac{1}{\cancel{2}\cos t \sin t} \cancel{2} dt$

$= \int \frac{1}{\frac{\sin t}{\cos t}\cdot\cos^2 t} \cdot dt = \int \frac{1}{\tan t}\cdot\frac{1}{\cos^2 t} dt = \begin{bmatrix} s = \tan t \\ ds = \frac{1}{\cos^2 t} dt \end{bmatrix} = \int \frac{1}{s} ds$

$= \ln|s| + C = \ln\left|\tan\frac{x}{2}\right| + C$

Ex. 12,27

$\int \frac{x\sqrt{x-1}}{1-\sqrt{x-1}} dx = \begin{bmatrix} t = \sqrt{x-1} \\ dx = 2t\,dt \end{bmatrix} \int \frac{(t^2+1)t}{1-t} 2t\,dt = 2\int \frac{t^4+t^2}{1-t} dt$

$\overset{s=1-t}{\underset{ds=-dt}{=}} -2\int \frac{(1-s)^4 + (1-s)^2}{s} =$ tillämpa binomialsatsen och lös

Ex. $\int \frac{x}{\sqrt{x^2+1}} dx \begin{bmatrix} t = x^2+1 \\ dt = 2x\,dx \end{bmatrix} = \frac{1}{2}\int \frac{1}{\sqrt{t}} dt = \frac{1}{2}\int t^{-\frac{1}{2}} dt = \frac{1}{\cancel{2}} \frac{t^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} + C$

$= \sqrt{x^2+1} + C$

Ex. $x^2\cos(x)^3 dx \overset{t=t^3}{\underset{dt=3x^2dx}{=}} \frac{1}{3}\int \cos t\, dt = \frac{1}{3}\sin t + C = \frac{1}{3}\sin(x^3) + C$

Ex. $\int x^2\sqrt{x-2}\, dx = \begin{bmatrix} s = x-2 \\ ds = dx \end{bmatrix} = \int (2+s)^2 s^{\frac{1}{2}} ds = \int (4+4s+s^2) s^{\frac{1}{2}} ds$

$= \int (4s^{\frac{1}{2}} + 4s^{\frac{3}{2}} + s^{\frac{5}{2}}) ds = 4\frac{s^{\frac{3}{2}}}{\frac{3}{2}} + 4\frac{s^{\frac{5}{2}}}{\frac{5}{2}} + \frac{s^{\frac{7}{2}}}{\frac{7}{2}}$ utveckla integral