

Föreläsning 4 LV 2

$$\int f(g(x)) g'(x) dx = \int_{t=g(x)} f(t) dt$$

Ex. $\int x\sqrt{x+1} dx$ $\begin{matrix} t=x+1 \\ dt=dx \end{matrix}$ $\int (t-1)t^{\frac{1}{2}} dt = \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$$= \frac{2}{5} t^{\frac{5}{2}} - \frac{2}{3} t^{\frac{3}{2}} + C = \frac{2}{5} (x+1)^{\frac{5}{2}} - \frac{2}{3} (x+1)^{\frac{3}{2}} + C$$

Alt. $\int x\sqrt{x+1} dx$ $\begin{matrix} s=\sqrt{x+1} \\ x=s^2-1 \\ dx=2s ds \end{matrix}$ $\int (s^2-1)s \cdot 2s ds = 2 \int (s^4 - s^2) ds$

$$= 2 \left(\frac{s^5}{5} - \frac{s^3}{3} \right) + C = 2 \left(\frac{(x+1)^{\frac{5}{2}}}{5} - \frac{(x+1)^{\frac{3}{2}}}{3} \right) + C$$

Sats. Formeln för partialintegration

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

$$\int f(x)g(x) dx = \int F'(x)g(x) dx = F(x)g(x) - \int F(x)g'(x) dx$$

Bevis av ① $\int (f(x)g(x))' dx = f(x)g(x) + C$

Men $\int (f'(x)g(x) + f(x)g'(x)) dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx = H_2$

som ger $\int f(x)g'(x) dx = f(x)g(x) + C - \int f'(x)g(x) dx$

Ex. $\int x e^x dx = \int x (e^x)' dx = x e^x - \int (x)' e^x dx = x e^x - e^x$

b) $\int x^2 e^x dx = \int x^2 (e^x)' dx = x^2 e^x - \int 2x e^x dx = x^2 e^x - \int 2x (e^x)' dx$
 $= x^2 e^x - (2x e^x - \int 2 e^x dx) = x^2 e^x - 2x e^x + 2 e^x + C$

Ex. 12, 12

$$\int \ln x dx = \int (x)' \cdot \ln x dx = x \cdot \ln x - \int x \cdot (\ln x)' dx = x \cdot \ln x - \int dx$$

$$= x \cdot \ln x - x + C$$

Ex. 12.14

Beräkna $I = \int e^{2x} \cos x \, dx =$

Lösning: $I = \int e^{2x} (\sin x)' \, dx = e^{2x} \sin x - \int \underbrace{e^{2x} \cdot 2}_{(e^{2x})'} \sin x \, dx$

$$= e^{2x} \sin x - 2 \int e^{2x} (-\cos x)' \, dx = e^{2x} \sin x - 2(e^{2x}(-\cos x) - \int e^{2x} \cdot 2(-\cos x) \, dx)$$

$$= e^{2x} \sin x + 2e^{2x} \cos x - 4 \int e^{2x} \cos x \, dx$$

$\underbrace{\hspace{10em}}_I$

$$I = e^{2x} \sin x + 2e^{2x} \cos x - 4I \quad \text{Lös ut } I!$$

4. Ex. $\int \frac{x}{x^4+1} \, dx \stackrel{t=x^2}{=} \frac{1}{2} \int \frac{1}{t^2+1} \, dt = \frac{1}{2} \arctan(t) + C$
 $dt = 2x \, dx$ $\underbrace{\hspace{10em}}_{(\arctan t)'}$

$$= \frac{1}{2} \arctan(x^2) + C$$

Problem: Hur beräknar man $\int \frac{p(x)}{q(x)} \, dx$?

\swarrow Polynom
 \nwarrow Polynom

$$\frac{p}{q} = \text{kvot} + \frac{\text{rest}}{q}$$

Lösningssmetod: (Partialbråksuppdelning)

Steg 1. Genom polynomdivision kan vi anta att täljarens grad < nämnarens grad

Steg 2. Faktorisera nämnaren $q(x)$ som produkt av polynom med grad 1. lr. 2.

Steg 3. Varje första grads faktor $(ax+b)^n$ ger $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_n}{(ax+b)^n}$

Varje "andragrads" faktor $(ax^2+bx+c)^n$ ger

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Steg 4. Bestäm alla dessa konstanter $A_1 \dots B_1 \dots$

Steg 5. Integrera varje term.

Ex. Beräkna $I = \int \frac{x^3+3x+1}{x^4+x^2} \, dx$

Lösning: Steg 1. Polynomdivision $\frac{x^3+3x+1}{x^4+x^2}$

$$I = \int \left(1 + \frac{-x^2+3x+1}{x^4+x^2} \right) dx = x + \int \frac{-x^2+3x+1}{x^4+x^2} dx$$

Steg 2 och 3.

$$\frac{-x^2+3x+1}{x^1+x^2} = \frac{-x^2+3x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

Steg 4. $\therefore -x^2+3x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$

div. $-x^2+3x+1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$

$$= (A+C)x^3 + (B+D)x^2 + Ax + B$$

som ger $\begin{cases} A+C=0 \\ B+D=-1 \\ A=3 \\ B=1 \end{cases}$ Ur detta kan vi enkelt bestämma A, B, C, D.
 $A=3 \quad B=1 \quad C=-3 \quad D=-2$

$$\therefore I = x + \int \left(\frac{3}{x} + \frac{1}{x^2} + \frac{-3x-2}{x^2+1} \right) dx = x + 3 \ln|x| + \frac{x^{-2+1}}{-2+1} - 3 \int \frac{x}{x^2+1} dx$$

* $\frac{x}{x^2+1} dx \stackrel{t=x^2+1}{dt=2x dx} \frac{1}{2} \int \frac{1}{t} dt = \frac{1}{2} \ln|t| + C_1 \quad -2 \int \frac{1}{x^2+1} dx \left. \vphantom{\int} \right\} \text{arctan } x + C$

$$= \frac{1}{2} \ln(x^2+1) + C_1$$

Föreläsning 5 LV 2 2015-11-12

12,4 Fortsättning

Ex. $\int \frac{x^2+1}{x(x+1)^2} dx$

Lösning: (Täljarens grad < nämnarens grad och faktorisering) ok!

Sätt $\frac{x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \quad \therefore x^2+1 = A(x+1)^2 + Bx(x+1) + Cx$

För $x=0$ gäller $1 = A + 0 + 0 \Leftrightarrow A=1$

För $x=-1$ gäller $2 = 0 + 0 - C \Leftrightarrow C=-2$

Nu kan man stoppa in vilket tal som helst för att få ut värdet av B.

För $x=1$ gäller $2 = 4A + 2B + C \Rightarrow B=0$

$$\therefore \int \frac{x^2+1}{x(x+1)} dx = \int \frac{1}{x} + \frac{-2}{(x+1)^2} dx = \ln|x| - 2 \int \frac{1}{(x+1)^2} dx$$

$$= \ln|x| + 2 \cdot \frac{1}{x+1} + C$$

↑
basbyte $x+1=t$

Ex. Tentauppgift 96p

$$\int \frac{\sqrt{x}}{1+\sqrt{x}} dx \stackrel{t=\sqrt{x}}{=} \int \frac{t}{1+t} 2t dt = \int \frac{2t^2}{1+t} dt$$

stoppar in i slutet

$$* \frac{t^2}{t+1} = t-1 + \frac{1}{t+1} *$$

ny integral $\int (t-1 + \frac{1}{t+1}) dt$

$$= \frac{t^2}{2} + t + \ln|t+1| + C$$

* stoppar in 2

$$\int \frac{\sqrt{x}}{1+\sqrt{x}} dx = t^2 - 2t + 2 \ln|t+1| + C_1 \quad \leftarrow C_1 \text{ då } C_1 = 2C$$

$$= x - 2\sqrt{x} + 2 \ln|\sqrt{x}+1| + C_1 \quad (\sqrt{x})^2 = x \quad \sqrt{x^2} = |x|$$

12,5 Sats

$$① \int f(\cos x) \sin x dx \stackrel{\left[\begin{array}{l} t = \cos x \\ dt = -\sin x dx \end{array} \right]}{=} - \int f(t) dt$$

$$② \int f(\sin x) \cos x dx = \int f(t) dt \stackrel{\left[\begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right]}{=}$$

$$\text{Ex. } \int \frac{\sin x}{2\cos x + 1} dx = \int \frac{dt}{2t+1} = -\frac{\ln|2t+1|}{2} + C$$

$$\text{Alternativ lösning: } \int \frac{\sin x}{2\cos x + 1} dx = \int \frac{dt}{2(-\sin x)dx} = -\frac{1}{2} \int \frac{dt}{t}$$

$$= -\frac{1}{2} \ln|t| + C = -\frac{1}{2} \ln|2\cos x + 1| + C$$

Ex. 12,23

$$\int \cos^5 x dx \stackrel{\text{udda tal}}{=} \int \cos^4 x \cdot \cos x dx = \int (1 - \sin^2 x)^2 \cos x dx = \int (1 - 2t^2 + t^4) dt$$

$$= t - 2\frac{t^3}{3} + \frac{t^5}{5} + C$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$$

Ex. 12,22

$$\int \sin^2 x dx \stackrel{\text{jämn}}{=} \int \frac{1 - \cos 2x}{2} dx = \frac{1}{2} x - \frac{1}{2} \sin 2x + C$$

$$\begin{aligned}
 \text{Ex. } \int \frac{1}{\sin x} dx &= \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{1 - \cos^2 x} dx = \left[\begin{array}{l} t = \cos x \\ dt = -\sin x dx \end{array} \right] \\
 &= - \int \frac{dt}{1 - t^2} = \int \frac{dt}{(t-1)(t+1)} = \frac{1}{2} \int \frac{(t+1) - (t-1)}{(t-1)(t+1)} dt \\
 &= \frac{1}{2} \int \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt = \frac{1}{2} (\ln |t-1| - \ln |t+1|) + C \\
 &= \frac{1}{2} (\ln |\cos x - 1| - \ln |\cos x + 1|) + C
 \end{aligned}$$

Metoden fungerar också för $\int \frac{1}{\cos x} dx$

Alternativ lösning

$$\begin{aligned}
 \int \frac{1}{\sin x} dx &= \int \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} dx = \left[\begin{array}{l} t = \frac{x}{2} \\ dt = \frac{1}{2} dx \end{array} \right] = \int \frac{1}{2 \cos t \sin t} 2 dt \\
 &= \int \frac{1}{\frac{\sin t}{\cos t} \cdot \cos^2 t} dt = \int \frac{1}{\tan t} \cdot \frac{1}{\cos^2 t} dt = \left[\begin{array}{l} s = \tan t \\ ds = \frac{1}{\cos^2 t} dt \end{array} \right] = \int \frac{1}{s} ds \\
 &= \ln |s| + C = \ln \left| \tan \frac{x}{2} \right| + C
 \end{aligned}$$

Ex. 12,27

$$\int \frac{x \sqrt{x-1}}{1 - \sqrt{x-1}} dx = \left[\begin{array}{l} t = \sqrt{x-1} \\ dx = 2t dt \end{array} \right] \int \frac{(t^2+1)t}{1-t} 2t dt = 2 \int \frac{t^3+t^2}{1-t} dt$$

$$\begin{array}{l} s = 1-t \\ ds = -dt \end{array} - 2 \int \frac{(1-s)^3 + (1-s)^2}{s} ds = \text{tillämpa binomialsatsen och lös}$$

$$\begin{aligned}
 \text{Ex. } \int \frac{x}{\sqrt{x^2+1}} dx &= \left[\begin{array}{l} t = x^2+1 \\ dt = 2x dx \end{array} \right] = \frac{1}{2} \int \frac{1}{\sqrt{t}} dt = \frac{1}{2} \int t^{-\frac{1}{2}} dt = \frac{1}{2} \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C \\
 &= \sqrt{x^2+1} + C
 \end{aligned}$$

$$\text{Ex. } \int x^2 \cos(x^3) dx = \left[\begin{array}{l} t = x^3 \\ dt = 3x^2 dx \end{array} \right] \int \cos t dt = \frac{1}{3} \sin t + C = \frac{1}{3} \sin(x^3) + C$$

$$\begin{aligned}
 \text{Ex. } \int x^2 \sqrt{x-2} dx &= \left[\begin{array}{l} s = x-2 \\ ds = dx \end{array} \right] = \int (2+s)^2 s^{\frac{1}{2}} ds = \int (4 + 4s + s^2) s^{\frac{1}{2}} ds \\
 &= \int (4s^{\frac{1}{2}} + 4s^{\frac{3}{2}} + s^{\frac{5}{2}}) ds = 4 \frac{s^{\frac{3}{2}}}{\frac{3}{2}} + 4 \frac{s^{\frac{5}{2}}}{\frac{5}{2}} + \frac{s^{\frac{7}{2}}}{\frac{7}{2}} \quad \text{utveckla integral}
 \end{aligned}$$