Föreläsning 2015-11-23

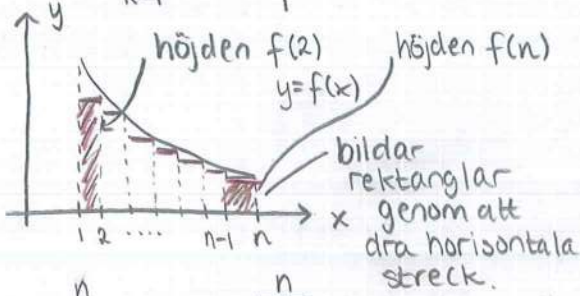
13,7 Uppskattning av summor

Sats. Antag att $f(x)$ är kontinuerlig, positiv och avtagande i $[1,n]$

Då gäller:

$$\int_1^n f(x) + f(n) \leq \sum_{k=1}^{n} f(k) \leq \int_1^n f(x)\,dx + f(1)$$

Bevis:



$$\int_1^n f(x)dx = \text{arean under } f(x)$$

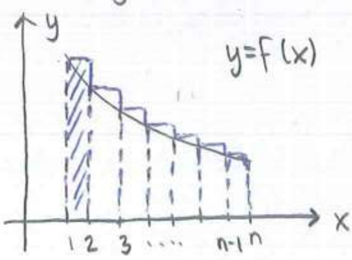
$\geq$ arean av trappfunktionen

$$= f(2)(2-1) + f(3)(3-2) + \ldots + f(n)(n-(n-1))$$

$$= \sum_{k=1}^{n} f(k) - f(1)$$

dvs. $$\int_1^n f(x)dx \geq \sum_{k=1}^{n} f(k) - f(1)$$

Dock gäller även



$$\therefore \int_1^n f(x)dx = \text{arean under } f(x) \leq \text{arean under trappfunktionen.}$$

$$= f(1)\cdot 1 + f(2)\cdot 1 + \ldots + f(n-1)\cdot 1$$

$$= \sum_{k=1}^{n} f(k) - f(n)$$

dvs. $$\int_1^n f(x)dx + f(n) \leq \sum_{k=1}^{n} f(k)$$

Ex. Uppskatta $\sum_{k=1}^{n} \frac{1}{k}$

Lösning: Skriv $\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} f(k)$ där $f(x) = \frac{1}{x}$

är kontinuerlig, positiv, avtagande i $[1,n]$

Detta stämmer enligt sats.

$$\int_1^n f(x)dx + f(n) \leq \sum_{k=1}^{n} f(k) \leq \int_1^n f(x)dx + f(1)$$

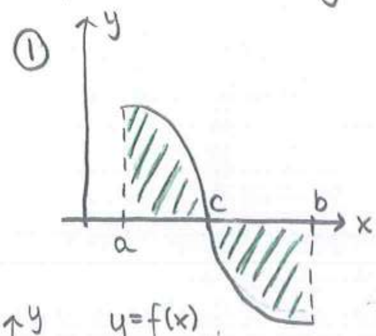
dvs. $$\underbrace{\int_1^n \frac{1}{x}dx}_{\ln n} + \frac{1}{n} \leq \sum_{k=1}^{n} \frac{1}{k} \leq \underbrace{\int_1^n \frac{1}{x}dx}_{\ln n} + \frac{1}{1}$$

som ger $\frac{1}{n} \leq \sum_{k=1}^{n} \frac{1}{k} - \ln n \leq 1 \quad \therefore \sum_{k=1}^{n} \frac{1}{k} \approx \ln n$

Kapitel 14

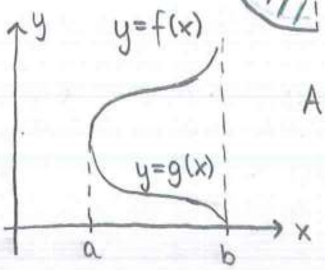
14.1 Areaberäkning

(1)



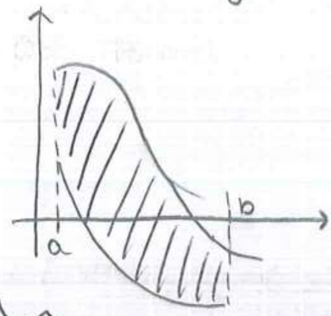
$$\text{Totala arean} = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

(2)

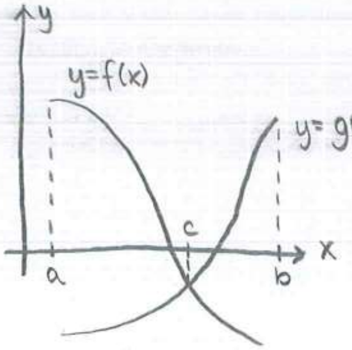


$$\text{Arean} = \int_a^b (f(x)+g(x))\,dx$$

OBS! Formel 2 gäller även för



(3)



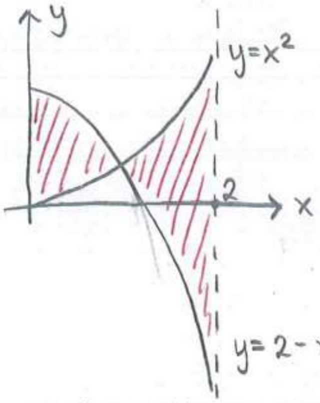
Totala arean

$$= \int_a^c (f(x)-g(x))\,dx + \int_c^b (g(x)-f(x))\,dx$$

14.2 Beräkna arean av det begränsade området, som begränsas av kurvorna

$y=x^2, \quad y=2-x^2, \quad x=0, \quad x=2$

Lösning:



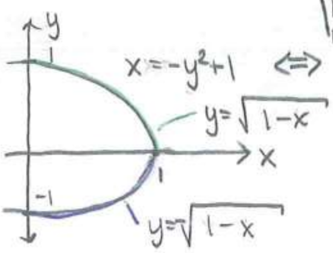
Skärningspunkten (x,y)

$$= \begin{cases} y=x^2 \\ y=2-x^2 \end{cases} \quad \text{som ger} \quad \begin{array}{l} x^2=1 \\ x=1 \end{array}$$

$$\text{Den sökta arean} = \int_0^1 \left((2-x^2)-x^2\right)dx$$

$$+ \int_1^2 \left(x^2-(2-x^2)\right)dx = \dots = 4$$

Ex.



$x=-y^2+1 \iff y=\pm\sqrt{1-x}$

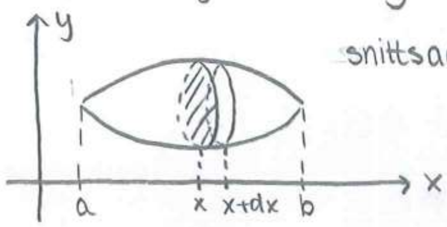
$y=\sqrt{1-x}$

$y=-\sqrt{1-x}$

$$\text{Arean} = \int_0^1 \left(\sqrt{1-x} - (-\sqrt{1-x})\right)dx =$$

$$2\int_0^1 \sqrt{1-x}\,dx = 2\left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1 = \frac{4}{3}$$

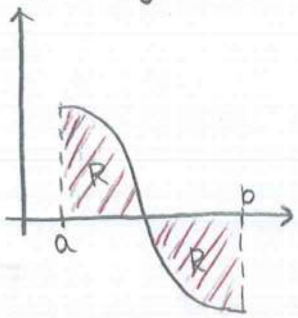
14,2. Volymberäkning



snittsarea $A(x)$ kroppens volym $= \int_a^b A(x)\,dx$

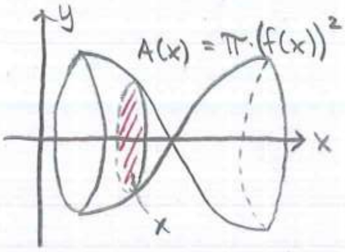
Rotationsvolym

①



Låt området R rotera runt x-axeln. Vi får en rotationskropp.

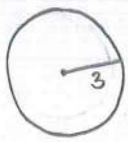
$A(x) = \pi \cdot (f(x))^2$



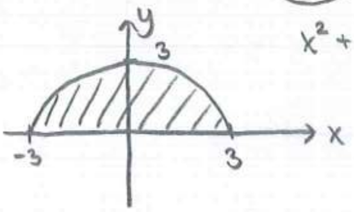
Rotationsvolym av området som begränsas av $y=f(x)$, x-axeln, $x=a$, $x=b$

Runt x-axeln ger $\int_a^b \pi f^2(x)\,dx$

Ex.



Beräkna volymen av klotet som har radien 3.

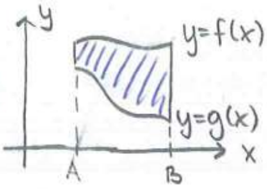


$x^2+y^2=3^2$, $y=\sqrt{9-x^2}$

Klotets volym = rotationsvolym

$= \pi \int \left(\sqrt{9-x^2}\right)^2 dx = \pi \int_{-3}^{3} (9-x^2)\,dx = \pi\, 9 \cdot 6 = 54\pi$

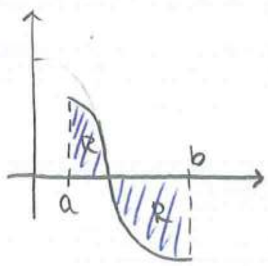
OBS!



Rotationkroppens volym runt x-axeln

$= \pi \int_a^b f(x)^2\,dx - \pi \int_a^b g(x)^2\,dx$

②



Låt området R rotera runt y-axeln.

Rotationsvolymen runt y-axeln $= 2\pi \int_a^b x\,|f(x)|\,dx$

obs. $a \geq 0$