Föreläsning 2015-12-10

11,2 $f(x) = P_n(x) + R_{n+1}(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$

$0 < \theta < 1$

Sats 11,3

① $e^x = 1 + x + \frac{x^2}{2!} + \dots$

② $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

③ $\frac{1}{1-x} = 1 + x + x^2 + \dots$

④ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$

⑤ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

⑥ $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots$

Ex. Bestäm $p_4(x)$ för $f(x) = \sin(x^2)$

Lösning: $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \Rightarrow \sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!}$

$= x^2 - \frac{x^6}{3!} + \dots \quad \therefore p_4(x) = x^2$

Lös $\int_0^1 \sin(x^2)\,dx \approx \int_0^1 p_4(x)\,dx$

$\Rightarrow \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

Ex. Bestäm $p_3(x)$ för $f(x) = \frac{1}{2+x^2}$

Lösning: Metod 1

$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!}x^3$ där $f(0) = \frac{1}{2}$

$f'(x) = -\frac{2x}{(2+x^2)^2} \Rightarrow f'(0) = 0$

$\therefore p_3(x) = \dots$

$f''(x) = \qquad \Rightarrow f''(0) = ?$

$f'''(x) = \qquad \Rightarrow f'''(0) = ?$

Metod 2

$f(x) = \frac{1}{2+x^2} = \frac{1}{2}\,\frac{1}{1+\frac{x^2}{2}} =$

$= \frac{1}{2}\,\frac{1}{1-\left(-\frac{x^2}{2}\right)} = \frac{1}{2}\left[1 + \left(-\frac{x^2}{2}\right) + \left(-\frac{x^2}{2}\right)^2 + \left(-\frac{x^2}{2}\right)^3 \dots\right]$

$= \frac{1}{2} - \frac{1}{4}x^2 + \frac{1}{8}x^4 - \frac{1}{16}x^6 + \dots$

$\therefore p_3(x) = \frac{1}{2} - \frac{1}{4}x^2$

11,8 Bestäm $P_3(x)$ till $f(x) = e^x \cdot \cos 2x$

Lösning Metod 1 Använd formeln för $P_3(x)$

Metod 2 $f(x) = e^x \cdot \cos 2x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots\right)$

$= 1 - \frac{(2x)^2}{2!} + x - \frac{(2x)^2}{2!}x + \frac{x^2}{2!}\cdot 1 + \frac{x^3}{3!}\cdot 1 + \dots$

$\therefore P_3(x) = 1 - 2x^2 + x - 2x^3 + \frac{1}{2}x^2 + \frac{x^3}{6} = 1 + x - \frac{3}{2}x^2 - \frac{11}{6}x^3$

Ex. $\lim_{x\to0} \frac{\sin x - x}{x(\cos x - 1)} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to0} \frac{(\sin x - x)'}{(x(\cos x - 1))'} = \lim_{x\to0} \frac{\cos x - 1}{\cos x - 1 + x(-\sin x)} \overset{\text{"}\frac{0}{0}\text{"}}{=}$

$\lim_{x\to0} \frac{(\cos x - 1)'}{(\cos x - 1 - x\sin x)'} = \lim_{x\to0} \frac{-\sin x}{-\sin x - \sin x - x\cos x} = \lim_{x\to0} \frac{\sin x}{2\sin x + x\cos x}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \frac{(\sin x)'}{(2\sin x + x\cos x)'} = \lim_{x\to0} \frac{\cos x}{2\cos x + \cos x + x(-\sin x)} = \frac{1}{2+1} = \frac{1}{3}$

Metod 2 $\lim_{x\to0} \frac{\sin x - x}{x(\cos x - 1)} = \lim_{x\to0} \frac{x - \frac{x^3}{3!} + \dots}{x\left(1 - \frac{x^2}{2!} + \dots - 1\right)} = \lim_{x\to0} \frac{-\frac{x^3}{3!} + \underbrace{\dots}_{x^4B(x)}}{-\frac{1}{2!}x^3 + \underbrace{\dots}_{x^4B_1(x)}}$

$= \lim_{x\to0} \frac{x^3\left(-\frac{1}{3!} + \dots\right)}{x^3\left(-\frac{1}{2!} + \dots\right)} = \frac{-\frac{1}{3!}}{-\frac{1}{2!}} = \frac{1}{3}$

Ex. 11,10. $\lim_{x\to0} \frac{e^{-2x^2} - \cos 2x}{x\cdot\sin x - x^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to0} \frac{1 + (-2x^2) + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \dots - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \dots\right)}{x\left(\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) - \cancel{x^2}}$

$\lim_{x\to0} \frac{2 - \frac{2}{3}x^4 + ?\,x^6 + \dots}{-\frac{x^4}{3!} + \frac{x^6}{5!} - \dots} = \lim_{x\to0} \frac{x^4\left(\frac{4}{3} + ?\,x^2 + \dots\right)}{x^4\left(-\frac{1}{6} + \frac{x^2}{5!} + \dots\right)} = \frac{\frac{4}{3}}{-\frac{1}{6}} = -8$

11,3 Maclaurinserier

Maclaurinutvecklingen ger $f(x) = \overbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k}^{P_n(x) \text{ om } R_{n+1}(x)\to 0 \text{ då } n\to\infty} + R_{n+1}(x) \xrightarrow{\text{då } n\to\infty} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$

Sats. Om $R_{n+1}(x) \to 0$ då $n \to \infty$ så är $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$ ← Maclaurinserien till $f(x)$.

$f(x) \approx P_n(x)$.

Sats. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ för alla $-\infty < x < \infty$

Bevis: $e^x = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}}_{R_{n+1}(x)} = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\theta x}x^{n+1}}{(n+1)!} \longrightarrow \sum_{k=0}^{\infty} \frac{x^k}{k!}$ om

$\frac{e^{\theta x}x^{n+1}}{(n+1)!} \to 0$ då $n \to \infty$ för varje fixt x., ty $\left|\frac{e^{\theta x}x^{n+1}}{(n+1)!}\right| \leq \frac{e^{|x|}|x|^{n+1}}{(n+1)!}$

$\boxed{|e^{\theta x}| \leq e^{|x|} \quad \theta x \leq |x| \quad 0 < \theta < 1}$

$e^{|x|}\frac{|x|}{1}\cdot\frac{|x|}{2}\cdots\frac{|x|}{n+1} \leq e^{|x|}\frac{|x|}{1}\cdot\frac{|x|}{2}\cdots\frac{|x|}{|x|+1}\cdot\frac{|x|}{|x|+2}\cdots\frac{|x|}{n+1}$

$\leqslant e^{|x|} \frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \frac{|x|}{3} \cdots \frac{|x|}{[|x|]}$ Vi behöver inte kunna beviset

$[a]$ är största heltal $\leqslant a$

Taylorutveckling av $f(x)$ runt $x=a$

$$\underbrace{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x)} + R_{n+1}(x,a)$$

$P_n(x)$ n:te grads taylorpolynomet av $f(x)$ i a

Ex. Bestäm 3:e grads Taylorpolynomet till $f(x) = \sin x$ i $x=2$

Lösning: 1) formel

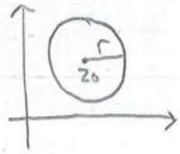
2) $f(x) = \sin x = \sin\left[(x-2)+2\right] = \sin(x-2)\cos 2 + \cos(x-2)\cdot\sin 2$

$= \left((x-2) - \frac{(x-2)^3}{3!} + \ldots\right)\cos 2 + \left(1 - \frac{(x-2)^2}{2!} + \ldots\right)\sin 2$

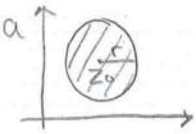
$= \underbrace{4 \text{ termer}}_{P_3(x)} + \ldots$

Repetition 1.

Kap. 6. 1) $|z-z_0| = r$ är en cirkel



2) $|z-z_0| < r$ är en cirkelskiva



3) $\left|\frac{z_1 \cdot z_2}{z_3 \cdot z_4}\right| = \frac{|z_1|\cdot|z_2|}{|z_3|\cdot|z_4|}$

ex. $\left|\frac{(1-i)3i}{2-i}\right| = \frac{|1-i|\,|3i|}{|2-i|} = \frac{\sqrt{2}\cdot 3}{\sqrt{5}}$

Ex. 4a) Visa att $\left|\frac{3z-1}{z-3}\right| = 1$ om $|z| = 1$

Lösning: $\left|\frac{3z-1}{z-3}\right| = \frac{|3z-1|}{|z-3|} = \frac{|3x-1+3yi|}{|(x+3)+yi|} = \frac{(3x-1)^2+9y^2}{\sqrt{(x+3)^2+y^2}}$

$= \frac{\sqrt{9x^2-6x+1+9y^2}}{\sqrt{x^2-6x+9+y^2}} = \sqrt{\frac{10-6x}{10-6x}} = 1$

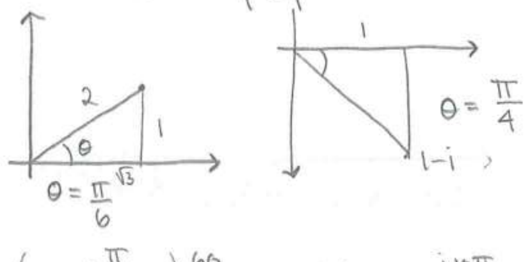
4) $e^{x+yi} = e^x e^{yi} = e^x(\cos y + i\sin y)$

5) Eulers formel $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ ty, $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-\theta i} = \cos\theta - i\sin\theta$

6) De Moivres formel $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ dvs. $(e^{i\theta})^n = e^{in\theta}$

Ex. Beräkna $\left(\frac{\sqrt{3}+i}{1-i}\right)^{60}$ (0,5p)

Lösning: $\frac{\sqrt{3}+i}{1-i}$



$r = |1-i| = \sqrt{2}$

$$\left(\frac{2e^{i\frac{\pi}{6}}}{\sqrt{2}e^{i(-\frac{\pi}{4})}}\right)^{60} = \left(2^{\frac{1}{2}}\right)^{60} \cdot \frac{e^{i10\pi}}{e^{(-i15\pi)}} = 2^{30} \cdot e^{i25\pi} = 2^{30}(-1+0) = -2^{30}$$

7) Kunna lösa andragradsekvationen

Lös $iz^2 + (2-2i)z - 4 = 0$ $\qquad z^2 + \frac{2(1-i)i}{-1} + \frac{4i}{-1} = 0$

$z^2 + \frac{2(1-i)}{i} - \frac{4}{i} = 0$ $\qquad \underbrace{z^2 - 2(1+i)z}_{p} \underbrace{+4i}_{q} = 0$

$(z-(1+i))^2 + 2i = 0$ $\qquad$ Sätt $x+yi = \sqrt{-2i}$ Då är $(x+yi)^2 = -2i$

$z = 1+i \pm \sqrt{-2i}$ $\qquad |x^2+y^2| = |2|$ och $x^2 + 2xyi - y^2 = -2i$

$\Rightarrow \begin{cases} x^2 - y^2 = 0 \\ 2xy = -2 \end{cases}$

① + ② $= 2x^2 = 2 \Rightarrow x = \pm 1$ $\qquad$ då $x = -1$ $\quad x+yi = -(1-i) = -1+i$

$x = 1$ $\quad x+yi = 1-i$

$z = 1+i \pm (1-i)$

8) $z^n = r_0 e^{i\theta_0}$ har n stycken olika rötter

$z_k = r^{\frac{1}{n}} e^{i\frac{\theta_0 + 2k\pi}{n}}$ där $k = 0, 1, 2 \ldots, n-1$

Ex. Lös $z^3 = 1 + \sqrt{3}i$ $\quad$ Lösning: $z_k = 2^{\frac{1}{3}} e^{i\frac{\frac{\pi}{3} + 2\pi k}{3}}$ för $k = 0, 1, 2$

9) $a_n z^n + \ldots + a_1 z + a_0 = 0$ med reella koefficienten $a_n \ldots a_0$ har en rot $z = \alpha + \beta i$ så även $z = \alpha - \beta i$

Kap 12

1) Primitiva funktioner sidan 281

2) Partiell integration

$$\int f \cdot g' \, dx = f \cdot g - \int f \cdot g' \, dx$$

3) $\int f(g(x)) \cdot g'(x) dx = \int f(t) dt$

Speciellt a) $\int h(x^{n+1}) \, x^n \, dx = \int h(t) \frac{dt}{n+1}$

b) $\int h(\sqrt{ax+b}) dx = \int h(t) \frac{2t}{a} dt$

c) $\int h(\cos x) \sin x \, dx = -\int h(t) \, dt$

4) Kunna partialbråksuppdelning