FÖRELÄSNING 19 — 1 mars

Imorgon räknar vi tenta 16-01-07

Idag exempel på residualkalkyl.

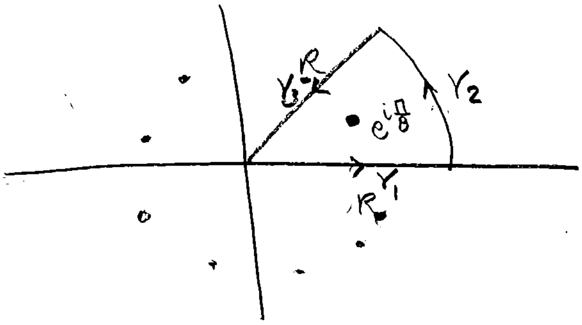
Ex. (jfr. 10:19)

Euler: $\displaystyle\int_0^{+\infty} \frac{x^{m-1}}{1+x^n} = \frac{\pi}{n\sin\left(\frac{m\pi}{n}\right)}$, $0<m<n$

Vi kollar på $m=1$, $n=8$:

$$\int_0^{+\infty} \frac{1}{1+x^8}\,dx$$

Vi använder kurvan:



γ_1: Intervallet $[0,R]$

γ_2: Del av cirkeln $|z|=R$, $0\le \arg z \le \frac{\pi}{4}$

γ_3: Linjestycket tillbaka i origo

"Vi har i själva verket 8 poler, men vi använder en kurva som bara täcker en av dem."

Vi använder $f(z) = \dfrac{1}{1+z^8}$

Residysatsen säger att: $\displaystyle\int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \int_{\gamma_3} f\,dz = i2\pi \operatorname*{Res}_{x=e^{i\frac{\pi}{8}}}(f)$

$$\int_{\gamma_1} f(z)\,dz = \int_0^R \frac{1}{1+x^8}\,dx \to \int_0^{+\infty} \frac{1}{1+x^8}\,dx$$

$$\Big|\int_{\gamma_2} f(z)\,dz\Big| \underset{\text{ML-olikheten}}{\le} \frac{CR}{R^8} \to 0 \text{ då } R\to+\infty$$

γ_3: $-\gamma_3$ har parametrisering $z = e^{i\frac{\pi}{4}}\cdot t$, $0\le t\le R$

Ty $\left(e^{i\frac{\pi}{4}}\right)^8 = e^{i2\pi} = 1$

$$\int_{\gamma_3} f(z)\,dz = -\int_{-\gamma_3} f(z)\,dz = -\int_0^R \frac{1}{1+\left(e^{i\frac{\pi}{4}}\cdot t\right)^8}\cdot e^{i\frac{\pi}{4}}\,dt = -e^{i\frac{\pi}{4}}\int_0^R \frac{1}{1+t^8}\,dt$$

$$\to -e^{i\frac{\pi}{4}}\int_0^{+\infty} \frac{1}{1+x^8}\,dx \Rightarrow$$

Så $\left(1-e^{i\frac{\pi}{4}}\right)\displaystyle\int_0^{+\infty}\frac{1}{1+x^8}\,dx = i2\pi \underset{z=e^{i\frac{\pi}{8}}}{\operatorname{Res}}(f) \underset{(*)}{=} i2\pi\cdot\frac{1}{8z^7}\Big|_{z=e^{i\frac{\pi}{8}}}$

(*) residyl regeln $\operatorname{Res}\frac{f}{g} = \frac{f(p)}{g'(p)}$

$= i2\pi\left(-\frac{e^{i\frac{\pi}{8}}}{8}\right)$ då $\frac{1}{8z^7} = \frac{z}{\underset{=-1}{8z^8}} = -\frac{z}{8}$

$= -\frac{i\pi}{4}e^{i\frac{\pi}{8}}$

Så $\displaystyle\int_0^{+\infty}\frac{1}{1+x^8}\,dx = -i\frac{\pi}{4}e^{i\frac{\pi}{8}}\cdot\frac{1}{1-e^{\frac{i\pi}{4}}} = -i\frac{\pi}{4}\frac{1}{e^{-i\frac{\pi}{8}}-e^{i\frac{\pi}{8}}} =$

$= \frac{\pi}{8}\cdot\frac{1}{\sin\left(\frac{\pi}{8}\right)}$

Ex 10.15, 10.16 samma integral olika metoder

$\displaystyle\int_{-\infty}^{\infty}\frac{x\sin x}{1+x^2}\,dx$ Egentligen: problem med konvergens som borde undersökas

$\displaystyle\lim_{R\to+\infty}\int_{-R}^{R}\frac{x\sin x}{1+x^2}\,dx$ ger principalvärde och bevisar inte att den är konvergent. Man bör egentligen studera

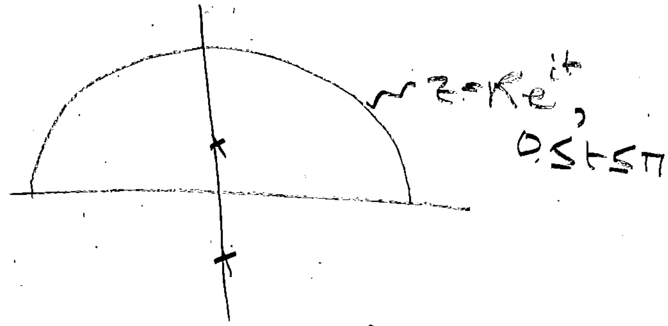
$\displaystyle\lim_{\substack{S\to+\infty\\ R\to+\infty}}\int_{-S}^{R}\frac{x\sin x}{1+x^2}\,dx$

$\Rightarrow$

Vad händer när vi integrerar längs C_R^+?

Med $f(z) = \dfrac{z\sin z}{1+z^2}$ blir

$$f(Re^{it}) = \frac{Re^{it}\sin(Re^{it})}{1+(Re^{it})^2}$$



Hur stor är $\sin(Re^{it})$?

$$\sin(Re^{it}) = \frac{e^{iRe^{it}} - e^{-iRe^{it}}}{2i} = \frac{e^{iR(\cos t + i\sin t)} - e^{-iR(\cos t + i \sin t)}}{2i}$$

$$= \frac{\overbrace{\boxed{e^{iR\cos t}}}^{|\,|=1} \cdot \overbrace{\boxed{e^{-R\sin t}}}^{\text{avtar exp.}} - \overbrace{\boxed{e^{-iR\cos t}}}^{|\,|=1} \cdot \overbrace{\boxed{e^{R\sin t}}}^{\text{växer exp}}}{2i}$$

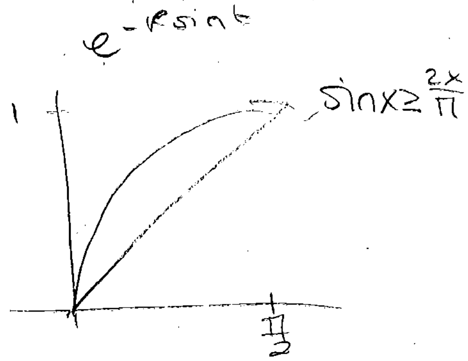
Så beloppet $|\sin(Re^{it})|$ växer exponentiellt med R!

Så $\left| \displaystyle\int_{C_R^+} \frac{z\sin z}{1+z^2}\,dz \right| \to +\infty$ då $R \to +\infty$

Idé: Låt istället $f(z) = \dfrac{z e^{iz}}{1+z^2}$

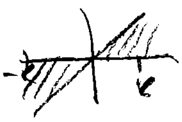
$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} \frac{x\cos x + ix\sin x}{1+x^2}\,dx$$

$$\int_{-\infty}^{+\infty} \frac{x\sin x}{1+x^2}\,dx = \operatorname{Im} \int_{-\infty}^{+\infty} \frac{z e^{iz}}{1+z^2}\,dz$$



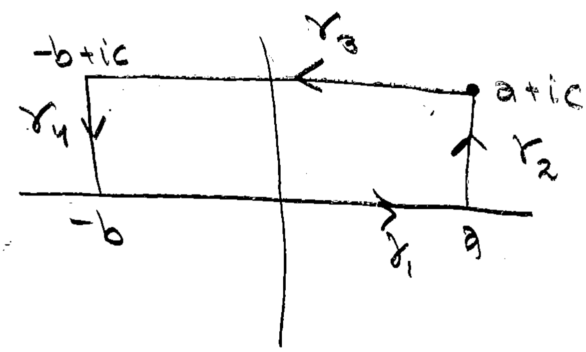
Jordans olikhet

$\displaystyle\int_{-R}^{R} x\,dx = 0$ för alla R



$\displaystyle\lim_{R\to+\infty} \int_{-R}^{R} x\,dx = 0$ men $\displaystyle\int_{-\infty}^{+\infty} x\,dx$ existerar ej

Låt $f(z) = \frac{z e^{iz}}{1+z^2}$



γ_1: $\int_{\gamma_1} f(z)\,dz = \int_{-b}^{a} \frac{x \sin x}{1+x^2}\, dx$

γ_2: $z = a+it$, $0 \le t \le c$

$$\left|\int_{\gamma_2} \frac{ze^{iz}}{1+z^2}\,dz\right| = \left|\int_0^c \frac{(a+it)\cdot e^{i(a+it)}}{1+(a+it)^2}\, i\,dt\right| \le \int_0^c \underbrace{\left|\frac{a+it}{1+(a+it)^2}\right|}_{\le \frac{D}{a}} e^{-t}\, dt$$

$$\le \frac{D}{a}\int_0^c e^{-t}\,dt = \frac{D}{a}\left(1-e^{-c}\right) \to 0 \text{ då } a\to\infty$$

PSS γ_4: $\left|\int_{\gamma_4} \frac{ze^{iz}}{1+z^2}\,dz\right| \le \frac{\tilde D}{b}\left(1-e^{-c}\right) \to 0$ då $b\to+\infty$

γ_3: $-\gamma_3$: $z = t+ic$ $\quad -b\le t\le a$

$$\left|\pm\int_{\gamma_3}\frac{(t+ic)e^{i(t+ic)}}{1+(t+ic)^2}\,dt\right| \le \int_{-b}^{a} \underbrace{\left|\frac{t+ic}{1+(t+ic)^2}\right|}_{\le \frac{\hat D}{c}} e^{-c}\,dt$$

$$\le \frac{\hat D}{c}(a+b)e^{-c} \underset{c=a+b}{=} \hat D e^{-(a+b)} \to 0 \text{ då } a \text{ eller } b \to +\infty$$

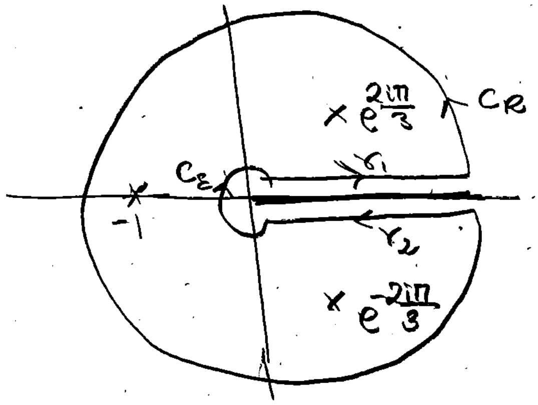
Residysatsen ger: $\int_{-\infty}^{+\infty}\frac{xe^{ix}}{1+x^2}\,dx = i2\pi \operatorname{Res}_{z=i}\left(\frac{ze^{iz}}{1+z^2}\right)$

$$\Rightarrow \int_{-\infty}^{+\infty}\frac{x\sin}{1+x^2}\,dx = \operatorname{Im}\int_{-\infty}^{+\infty}\frac{xe^{ix}}{1+x^2}\,dx = \operatorname{Im}\left(i2\pi\operatorname{Res}_{z=i}\left(\frac{ze^{iz}}{1+z^2}\right)\right) = \dots =$$

Ex. (10.17) $\sqrt{z}$

Vi beräknar $\int_{\gamma}^{+\infty} \frac{\sqrt{x}}{1+x^3}\,dx$ Principalgrenen

Vi kommer att välja den naturliga grenen



$f(z) = \frac{\sqrt{z}}{1+z^3}$ väldefinerad på γ.

Om $z = |z|e^{i\theta}$ så är $\sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}$

$0 < \theta < 2\pi$

Residysatsen: $\int_{\gamma} f(z)\,dz = i2\pi \left(\underset{e^{\frac{2\pi i}{3}}}{\operatorname{Res}}(f) + \underset{-1}{\operatorname{Res}}(f) + \underset{e^{-\frac{2\pi i}{3}}}{\operatorname{Res}}(f) \right)$

γ_1: $z = x > 0$ $(\varepsilon < x < R)$ Då vid närmar oss Re-axeln gör $\theta \to 0$.

$\sqrt{z}$ blir $\sqrt{x}$

$$\int_{\varepsilon}^{R} \frac{\sqrt{x}}{1+x^3}\,dx$$

$-\gamma_2$: Då vi närmar Re-axeln kommer $\theta \to 2\pi$.

Så $|z|^{1/2} e^{i\theta/2}$ närmar sig $|z|^{1/2} \cdot (-1)$

Vi får $\int_{\gamma_2} f(z)\,dz = -\int_{-\gamma_2} f(z)\,dz = -\int_{\varepsilon}^{R} \frac{\sqrt{x}(-1)}{1+x^3}\,dx =$

$$= \int_{\varepsilon}^{R} \frac{\sqrt{x}}{1+x^3}\,dx$$

$2\int_{0}^{+\infty} \frac{\sqrt{x}}{1+x^3}\,dx = i2\pi(\operatorname{Res} \ldots)$ (Visa att $\int_{C_R} \ldots \to 0$, $\int_{C_\varepsilon} \ldots \to 0$)