Lösningar kapitel 4

Endimensionell analys

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Lösta uppgifter

4.3

$$1+2+3+\ldots+99+100=\sum_{k=1}^{100} k=\frac{100(100+1)}{2}=5050$$

4.4

$$3+6+9+\ldots+96+99=3(1+2+3+\ldots+32+33)=3\sum_{k=1}^{33} k=3\frac{33(33+1)}{2}=1683$$

4.6

b)

$$\sum_{k=1}^{15}(3k+2)=\sum_{k=1}^{15}3k+\sum_{k=1}^{15}2=3\sum_{k=1}^{15}k+\sum_{k=1}^{15}2=3\frac{15(15+1)}{2}+15\cdot 2=390$$

4.7

a)

$$1+2+4+8+16+32=\sum_{k=0}^{5}2^k=\frac{2^{5+1}-1}{2-1}=2^6-1=63$$

b)

$$1-3+9-27+81-243=\sum_{k=0}^{5}(-3)^k=\frac{(-3)^{5+1}-1}{-3-1}=\frac{1-(-3)^6}{4}=-182$$

c)

$$2+1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{128}=2\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\ldots+\frac{1}{256}\right)=2\sum_{k=0}^{8}\left(\frac{1}{2}\right)^k$$

$$=2\frac{\left(\frac{1}{2}\right)^9-1}{\frac{1}{2}-1}=\frac{511}{128}$$

d)

$$e+e^2+e^3+\ldots+e^{10}=e(1+e+e^2+\ldots e^9)=e\sum_{k=0}^{9}e^k=e\frac{e^{10}-1}{e-1}$$

e)

$$1-x+x^2-x^3+\ldots-x^9=\sum_{k=0}^{9}(-x)^k=\frac{(-x)^{10}-1}{-x-1}=\frac{1-x^{10}}{1+x}$$

4.8

a)

$$\sum_{k=0}^{10} 3 \cdot 2^k = 3\sum_{k=0}^{10} 2^k = 3\frac{2^{11}-1}{2-1} = 3(2^{11}-1)$$

b)

$$\sum_{k=1}^{10} 3 \cdot 2^k = 3\sum_{k=0}^{9} 2^{k+1} = 3\sum_{k=0}^{9} 2^k 2 = 6\sum_{k=0}^{9} 2^k = 6\frac{2^{10}-1}{2-1} = 6(2^{10}-1)$$

4.9

a)

$$\sum_{k=0}^{n} 3 \cdot 2^{-k} = 3\sum_{k=0}^{n} 2^{-k} = 3\sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = 3\frac{\left(\frac{1}{2}\right)^{n+1}-1}{\frac{1}{2}-1} = 6\left(1 - \frac{1}{2^{n+1}}\right)$$

b)

$$\sum_{k=1}^{n} e^{-k} = \sum_{k=0}^{n-1} e^{-k-1} = e^{-1}\sum_{k=0}^{n-1} e^{-k} = e^{-1}\sum_{k=0}^{n-1} \left(\frac{1}{e}\right)^k = e^{-1}\frac{(e^{-1})^n - 1}{e^{-1}-1}$$

$$= \frac{1}{e}\frac{e^{-n}-1}{\frac{1}{e}-1} = \frac{e^{-n}-1}{1-e} = \frac{1-e^{-n}}{e-1}$$

c)

$$\sum_{n=0}^{100} 1000 \cdot 1.05^n = 1000\sum_{n=0}^{100} 1.05^n = 1000\frac{1.05^{101}-1}{1.05-1} = 20000(1.05^{101}-1)$$

4.11

$$\sum_{k=0}^{11} 2000 \cdot 1.02^k = 2000\sum_{k=0}^{11} 1.02^k = 2000\frac{1.02^{12}-1}{1.02-1} = 10^5(1.02^{12}-1) \approx 26800$$

4.13

a)

$$7! = 7 \cdot 6 \cdot 5 \cdot \ldots \cdot 2 \cdot 1 = 5040$$

b)

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3!} = 7 \cdot 5 = 35$$

c)

$$\binom{1001}{999} = \frac{1001!}{999!(1001-999)!} = \frac{1001!}{999! \cdot 2!} = \frac{1001 \cdot 1000}{2!} = 1001 \cdot 500 = 500500$$

4.14

a)

$$(a+b)^2 = a^2 + 2ab + b^2$$

b)

$$(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2) = a^3+2a^2b+ab^2+a^2b+2ab^2+b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

c)

$$\begin{aligned}(a+b)^4 &= (a+b)(a+b)^3 = (a+b)(a^3+3a^2b+3ab^2+b^3)\\ &= a^4+3a^3b+3a^2b^2+ab^3+a^3b+3a^2b^2+3ab^3+b^4\\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{aligned}$$

4.15

Vi kan använda oss av resultatet i 4.14

a)

$$(1+x)^3 = 1^3 + 3 \cdot 1^2 \cdot x + 3 \cdot 1 \cdot x^2 + x^3 = x^3 + 3x^2 + 3x + 1$$

b)

$$(3-2x)^3 = 3^3 + 3 \cdot 3^2 \cdot (-2x) + 3 \cdot 3 \cdot (-2x)^2 + (-2x)^3 = 27 - 54x + 36x^2 - 8x^3$$

c)

$$(1+x)^4 = 1^4 + 4 \cdot 1^3 \cdot x + 6 \cdot 1^2 \cdot x^2 + 4 \cdot 1 \cdot x^3 + x^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$$

4.16

Enligt binomialsatsen är den

$$\binom{15}{2} 1^2 = \binom{15}{2} = \frac{15!}{2!(15-2)!} = \frac{15 \cdot 14}{2!} = 15 \cdot 7 = 105$$

4.17

Enligt binomialsatsen är den

$$-\binom{8}{3} 3^5 = -243 \binom{8}{3} = -243 \frac{8!}{3! \cdot 5!} = -243 \cdot 8 \cdot 7 = -243 \cdot 56 = -13608$$