Lösningar kapitel 2

Endimensionell analys

Fabian Ågren, π11

Lösta uppgifter

2.1

a)

$$(x+3)(x-3)-(x+3)^2 = x^2-3^2-(x^2+2x\cdot 3+3^2) = x^2-9-(x^2+6x+9)$$
$$= x^2-9-x^2-6x-9 = -6x-18$$

c)

$$(3x+5)^2-(3x-5)^2 = (3x)^2+2\cdot 3x\cdot 5+5^2-(3x-5)^2 = 9x^2+30x+25-(3x+5)^2$$
$$= 9x^2+30x+25-\left((3x)^2+2\cdot 3x\cdot(-5)+(-5)^2\right)$$
$$= 9x^2+30x+25-(9x^2-30x+25) = 60x$$

2.2

$$(a-b)^3 = (a-b)^2(a-b) = (a^2-2ab+b^2)(a-b) = a^3-2a^2b+ab^2-a^2b+2ab^2-b^3$$
$$= a^3-3a^2b+3ab^2-b^3$$

2.3

$$(a-b)(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)(a^{16}+b^{16})$$
$$= (a^2-b^2)(a^2+b^2)(a^4+b^4)(a^8+b^8)(a^{16}+b^{16})$$
$$= (a^4-b^4)(a^4+b^4)(a^8+b^8)(a^{16}+b^{16})$$
$$= (a^8-b^8)(a^8+b^8)(a^{16}+b^{16})$$
$$= (a^{16}-b^{16})(a^{16}+b^{16})$$
$$= a^{32}-a^{32}$$

2.5

a)

$$\frac{1}{7}-\left(\frac{15}{14}+\frac{1}{2}\right) = \frac{1}{7}-\left(\frac{15}{14}+\frac{7}{14}\right) = \frac{1}{7}-\frac{22}{14} = \frac{1}{7}-\frac{11}{7} = -\frac{10}{7}$$

2.6

a)

$$\frac{1}{60}+\frac{1}{108}-\frac{1}{72} = \frac{108}{6480}+\frac{60}{6480}-\frac{1}{72} = \frac{168}{6480}-\frac{1}{72} = \frac{168}{6480}-\frac{90}{6480} = \frac{78}{6480} = \frac{13}{1080}$$

b)

$$\frac{3}{4}-\frac{5}{6}+\frac{1}{9} = \frac{18}{24}-\frac{20}{24}+\frac{1}{9} = -\frac{2}{24}+\frac{1}{9} = -\frac{18}{216}+\frac{24}{216} = \frac{6}{216} = \frac{1}{36}$$

2.7

a)

$$\frac{\frac{a}{2}}{\frac{a}{4}} = \frac{a}{2} \cdot \frac{4}{a} = 2$$

c)

$$\frac{\frac{14a}{a+2}}{\frac{7}{6a+12}} = \frac{14a}{a+2} \cdot \frac{6a+12}{7} = \frac{2a(6a+12)}{a+2} = \frac{12a(a+2)}{a+2} = 12a$$

2.8

a)

$$\frac{\frac{3}{5x} - \frac{x}{15}}{\frac{1}{x} - \frac{1}{3}} = \frac{\frac{45}{75x} - \frac{5x^2}{75x}}{\frac{3}{3x} - \frac{x}{3x}} = \frac{\frac{45-5x^2}{75x}}{\frac{3-x}{3x}} = \frac{3x(45-5x^2)}{75x(3-x)} = \frac{45-5x^2}{25(3-x)} = \frac{5(9-x^2)}{25(3-x)}$$

$$= \frac{9-x^2}{5(3-x)} = \frac{(3+x)(3-x)}{5(3-x)} = \frac{x+3}{5}$$

b)

$$\frac{x^2+1}{1+\frac{1}{x^2}} = \frac{x^2+1}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \frac{x^2+1}{\frac{x^2+1}{x^2}} = \frac{x^2(x^2+1)}{x^2+1} = x^2$$

c)

$$\frac{\frac{1}{x} - \frac{1}{y}}{\frac{x^2-y^2}{(xy)^2}} = \frac{\frac{-(x+y)}{xy}}{\frac{x^2-y^2}{(xy)^2}} = -\frac{(xy)^2(x+y)}{xy(x^2-y^2)} = -\frac{xy(x+y)}{x^2-y^2}$$

$$= -\frac{xy(x+y)}{(x+y)(x-y)} = -\frac{xy}{x+y}$$

2.9

a)

$$\frac{\frac{x}{y} - \frac{y}{x}}{\frac{x}{y} + \frac{y}{x} - 2} = \frac{\frac{x^2-y^2}{xy}}{\frac{x^2+y^2}{xy} - 2} = \frac{\frac{x^2-y^2}{xy}}{\frac{x^2-2xy+y^2}{xy}} = \frac{\frac{x^2-y^2}{xy}}{\frac{(x-y)^2}{xy}} = \frac{xy(x^2-y^2)}{xy(x-y)^2} = \frac{x^2-y^2}{(x-y)^2}$$

$$= \frac{(x+y)(x-y)}{(x-y)^2} = \frac{x+y}{x-y}$$

b)

$$\frac{\frac{16x^4}{81}-y^4}{\frac{2x}{3}+y}=\frac{\frac{16x^4-81y^4}{81}}{\frac{2x+3y}{3}}=\frac{3(16x^4-81y^4)}{81(2x+3y)}=\frac{16x^4-81y^4}{27(2x+3y)}$$
$$=\frac{(4x^2+9y^2)(4x^2-9y^2)}{27(2x+3y)}=\frac{(4x^2+9y^2)(2x+3y)(2x-3y)}{27(2x+3y)}$$
$$=\frac{(4x^2+9y^2)(2x-3y)}{27}=\frac{8x^3-12x^2y+18xy^2-27y^3}{27}$$

2.10

a)

$$\frac{1}{R}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{3}{6}+\frac{2}{6}+\frac{1}{4}=\frac{5}{6}+\frac{1}{4}=\frac{20}{24}+\frac{6}{24}=\frac{26}{24}=\frac{13}{12}$$
$$\iff R=\frac{12}{13}$$

b)

$$\frac{1}{3}=\frac{1}{R_1}+\frac{1}{5}\iff\frac{1}{R_1}=\frac{1}{3}-\frac{1}{5}\iff\frac{1}{R_1}=\frac{5}{15}-\frac{3}{15}\iff\frac{1}{R_1}=\frac{2}{15}\iff R_1=\frac{15}{2}$$

2.11

$$\frac{1}{600}+\frac{1}{b}=\frac{1}{100}\iff\frac{1}{b}=\frac{1}{100}-\frac{1}{600}\iff\frac{1}{b}=\frac{6}{600}-\frac{1}{600}$$
$$\iff\frac{1}{b}=\frac{5}{600}\iff\frac{1}{b}=\frac{1}{120}\iff b=120$$

2.13

$$\frac{3+\sqrt{5}}{2+\sqrt{5}}=\frac{(3+\sqrt{5})(2-\sqrt{5})}{(2+\sqrt{5})(2-\sqrt{5})}=\frac{6-3\sqrt{5}+2\sqrt{5}-5}{2^2-(\sqrt{5})^2}=\frac{1-\sqrt{5}}{4-5}=\sqrt{5}-1$$

2.14

a)

$$\frac{1+2\sqrt{2}}{3-\sqrt{2}}=\frac{(1+2\sqrt{2})(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}=\frac{3+\sqrt{2}+6\sqrt{2}+4}{9-2}=\frac{7+7\sqrt{2}}{7}=1+\sqrt{2}$$

c)

$$\frac{2}{\sqrt{x+1}+\sqrt{x-1}}=\frac{2\left(\sqrt{x+1}-\sqrt{x-1}\right)}{\left(\sqrt{x+1}+\sqrt{x-1}\right)\left(\sqrt{x+1}-\sqrt{x-1}\right)}$$

observera konjugatregeln i nämnaren

$$=\frac{2\left(\sqrt{x+1}-\sqrt{x-1}\right)}{x+1-(x-1)}=\frac{2\left(\sqrt{x+1}-\sqrt{x-1}\right)}{2}=\sqrt{x+1}-\sqrt{x-1}$$

2.15

a)

$$\sqrt{12}-\sqrt{3}=\sqrt{4\cdot 3}-\sqrt{3}=\sqrt{4}\sqrt{3}-\sqrt{3}=2\sqrt{3}-\sqrt{3}=\sqrt{3}(2-1)=\sqrt{3}$$

b)

$$\frac{\sqrt{42}}{\sqrt{6}}=\frac{\sqrt{6\cdot 7}}{\sqrt{6}}=\frac{\sqrt{6}\sqrt{7}}{\sqrt{6}}=\sqrt{7}$$

c)

$$\sqrt{3}\sqrt{12}=\sqrt{3}\sqrt{3\cdot 4}=\sqrt{3}\sqrt{3}\sqrt{4}=3\sqrt{4}=3\cdot 2=6$$

2.16

a)

$$\frac{\sqrt{162}+\sqrt{98}}{\sqrt{50}+\sqrt{2}}=\frac{\sqrt{81\cdot 2}+\sqrt{49\cdot 2}}{\sqrt{25\cdot 2}+\sqrt{2}}=\frac{9\sqrt{2}+7\sqrt{2}}{5\sqrt{2}+\sqrt{2}}=\frac{\sqrt{2}(9+7)}{\sqrt{2}(5+1)}=\frac{16}{6}=\frac{8}{3}$$

b)

$$\frac{\sqrt{(-4)^2}}{\sqrt{4^2}}=\frac{\sqrt{16}}{\sqrt{16}}=1$$

c)

$$\left(\sqrt{12}-\frac{1}{\sqrt{3}}\right)^2=(\sqrt{12})^2-2\frac{\sqrt{12}}{\sqrt{3}}+\left(\frac{1}{\sqrt{3}}\right)^2=12-2\frac{\sqrt{3}\sqrt{4}}{\sqrt{3}}+\frac{1}{3}=12-4+\frac{1}{3}$$
$$=8+\frac{1}{3}=\frac{24}{3}+\frac{1}{3}=\frac{25}{3}$$

d)

Använd konjugatregeln

$$\left((\sqrt{x}+\sqrt{y})+\sqrt{x+y}\right)\left((\sqrt{x}+\sqrt{y})-\sqrt{x+y}\right)=(\sqrt{x}+\sqrt{y})^2-\sqrt{x+y}^2$$
$$=x+2\sqrt{x}\sqrt{y}+y-(x+y)=2\sqrt{x}\sqrt{y}=2\sqrt{xy}$$

2.18

b)

$$2^7\cdot 2^{-3}=2^{7+(-3)}=2^4$$

d)

$$\frac{3^7}{3^3}=3^{7-3}=3^4$$

2.19

a)

$$3^5 \cdot 10^5 \cdot 3^{-3} \cdot 10^3 = 3^{5+(-3)} \cdot 10^{5+3} = 3^2 \cdot 10^8 = 9 \cdot 10^8$$

b)

$$\frac{2^8 \cdot 5^6}{2^6 \cdot 5^5} = \frac{2^8}{2^6} \cdot \frac{5^6}{5^5} = 2^{8-6} \cdot 5^{6-5} = 2^2 \cdot 5 = 4 \cdot 5 = 20$$

2.20

b)

$$(a^3)^{-0.5} \cdot (a^{-5})^{-0.3} = a^{3\cdot(-0.5)} \cdot a^{-5\cdot(-0.3)} = a^{-1.5} \cdot a^{1.5} = a^{-1.5+1.5} = a^0 = 1$$

d)

$$\frac{x \cdot x^{-1.6} \cdot x^{0.2}}{x^{-1.4}} = \frac{x^{1+(-1.6)+0.2}}{x^{-1.4}} = \frac{x^{-0.4}}{x^{-1.4}} = x^{-0.4-(-1.4)} = x^1 = x$$

2.21

a)

$$\frac{3^2 \cdot 2^4}{6^3} = \frac{3^2 \cdot 2^4}{(3 \cdot 2)^3} = \frac{3^2 \cdot 2^4}{3^3 \cdot 2^3} = \frac{2}{3}$$

b)

$$\left(\frac{1}{4}\right)^{-\frac{1}{2}} = \frac{1}{\left(\frac{1}{4}\right)^{\frac{1}{2}}} = \frac{1}{\sqrt{\frac{1}{4}}} = \frac{1}{\frac{\sqrt{1}}{\sqrt{4}}} = \frac{1}{\frac{1}{2}} = 2$$

c)

$$(\sqrt{64})^{\frac{2}{3}} = 8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$$

d)

$$\left(\frac{1}{3}\right)^{-1} = \frac{1}{\frac{1}{3}} = 3$$

e)

$$2^{(2^3)} = 2^8 = 256$$

f)

$$\left(2^2\right)^3 = 2^{2\cdot 3} = 2^6 = 64$$

2.23

a)

$$\frac{a^{3.3}a^{-2.1}}{a^{0.8}} = \frac{a^{3.3-2.1}}{a^{0.8}} = \frac{a^{1.2}}{a^{0.8}} = a^{1.2-0.8} = a^{0.4}$$

b)

$$\frac{a\sqrt{a}}{\sqrt[3]{a^2}} = \frac{aa^{\frac{1}{2}}}{(a^2)^{\frac{1}{3}}} = \frac{a^{1+\frac{1}{2}}}{a^{\frac{2}{3}}} = \frac{a^{\frac{3}{2}}}{a^{\frac{2}{3}}} = a^{\frac{3}{2}-\frac{2}{3}} = a^{\frac{5}{6}}$$

2.24

$$\frac{(2^6 \cdot 5^5)^3}{(2^4 \cdot 5^2)^4} \cdot \frac{2^{18}}{5^7} = \frac{2^{18} \cdot 5^{15} \cdot 2^{18}}{2^{16} \cdot 5^8 \cdot 5^7} = \frac{2^{36} \cdot 5^{15}}{2^{16} \cdot 5^{15}}$$

$$= \frac{2^{36}}{2^{16}} = 2^{20} = (2^4)^5 = (2^2 \cdot 2^2)^5 = (4^2)^5$$

jämförelse med VL ger att $x = 2$.

2.25

a)

Använd konjugatregeln

$$x^2 - 1 = (x+1)(x-1)$$

c)

Använd konjugatregeln

$$x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$$

d)

Använd kvadreringsregeln

$$x^2 - 2x + 1 = (x-1)^2$$

h)

Bryt ut och använd kvadreringsregeln

$$a^2b + 2ab^2 + b^3 = b(a^2 + 2ab + b^2) = b(a+b)^2$$

i)

Bryt ut ab och använd kvadreringsregeln

$$a^3b - 2a^2b^2 + ab^3 = ab(a^2 - 2ab + b^2) = ab(a-b)^2$$

2.27

a)

$$x^2 - 7x + xy - 7y = x(x-7) + y(x-7) = (x-7)(x+y)$$

c)

Använd konjugatregeln i sista steget

$$\begin{aligned} x^2y + 2x^2 - y - 2 &= x^2(y+2) - y - 2 = x^2(y+2) - (y+2) \\ &= (y+2)(x^2-1) = (y+2)(x+1)(x-1) \end{aligned}$$

f)

Börja med konjugatregeln i första steget. Använd sen kvadreringsregeln på de två termerna i sista steget

$$(x^2+y^2)^2 - (2xy)^2 = (x^2+y^2+2xy)(x^2+y^2-2xy) = (x+y)^2(x-y)^2$$

2.28

a)

$$x^2 + 6x + 7 = x^2 + 6x + 9 - 9 + 7 = (x+3)^2 - 2$$

b)

$$\begin{aligned} x^2 - 7x + 13 &= x^2 - 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 13 = \left(x - \frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 13 \\ &= \left(x - \frac{7}{2}\right)^2 - \frac{49}{4} + 13 = \left(x - \frac{7}{2}\right)^2 - \frac{49}{4} + \frac{52}{4} = \left(x - \frac{7}{2}\right)^2 + \frac{3}{4} \end{aligned}$$

d)

$$x^2 + 5x = x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4}$$

2.29

Börja med att utveckla högerledet

$$(x+b)^2 + c = x^2 + 2bx + b^2 + c$$

Jämför nu med vänsterledet. Vi får då ett ekvationssystem

$$\begin{cases} 2b = a \\ b^2 + c = 0 \end{cases} \iff \begin{cases} b = \frac{a}{2} \\ c = -b^2 \end{cases} \iff \begin{cases} b = \frac{a}{2} \\ c = -\frac{a^2}{4} \end{cases}$$

2.30

a)

Använd konjugatregeln i täljaren

$$\frac{4x^2-4}{2x+2} = \frac{(2x+2)(2x-2)}{2x+2} = 2x-2$$

c)

$$\frac{1}{x^2} - \frac{1}{y^2} + \frac{x^2-y^2}{(xy)^2} = \frac{y^2}{x^2y^2} - \frac{x^2}{x^2y^2} + \frac{x^2-y^2}{x^2y^2} = \frac{y^2-x^2+x^2-y^2}{x^2y^2} = \frac{0}{x^2y^2} = 0$$

2.31

a)

$$\begin{aligned}\frac{2}{3x+9} + \frac{x}{x^2-9} - \frac{1}{2x-6} &= \frac{2}{3(x+3)} + \frac{x}{(x+3)(x-3)} - \frac{1}{2(x-3)} \\ &= \frac{4(x-3)+6x-3(x+3)}{6(x+3)(x-3)} = \frac{4x-12+6x-3x-9}{6(x+3)(x-3)} \\ &= \frac{7x-21}{6(x+3)(x-3)} = \frac{7(x-3)}{6(x+3)(x-3)} \\ &= \frac{7}{6(x+3)}\end{aligned}$$

2.32

a)

$$\begin{aligned}\frac{3x-y}{x^2-2xy+y^2} - \frac{2}{x-y} - \frac{2y}{(x-y)^2} &= \frac{3x-y}{(x-y)^2} - \frac{2}{x-y} - \frac{2y}{(x-y)^2} \\ &= \frac{3x-y}{(x-y)^2} - \frac{2(x-y)}{(x-y)^2} - \frac{2y}{(x-y)^2} = \frac{3x-y-2(x-y)-2y}{(x-y)^2} \\ &= \frac{x-y}{(x-y)^2} = \frac{1}{x-y}\end{aligned}$$

b)

$$\begin{aligned}\frac{a}{a^2+4ab+4b^2} + \frac{2b}{a^2-4b^2} &= \frac{a}{(a+2b)^2} + \frac{2b}{(a+2b)(a-2b)} \\ &= \frac{a(a-2b)}{(a+2b)^2(a-2b)} + \frac{2b(a+2b)}{(a+2b)^2(a-2b)} = \frac{a(a-2b)+2b(a+2b)}{(a+2b)^2(a-2b)} \\ &= \frac{a^2+4b^2}{(a+2b)^2(a-2b)}\end{aligned}$$

2.33

$$\frac{1}{(x-1)(x-2)} + \frac{1}{(x-1)(x-3)} + \frac{1}{(x-2)(x-3)}$$
$$= \frac{x-3}{(x-1)(x-2)(x-3)} + \frac{x-2}{(x-1)(x-2)(x-3)} + \frac{x-1}{(x-1)(x-2)(x-3)}$$
$$= \frac{x-3+x-2+x-1}{(x-1)(x-2)(x-3)} = \frac{3x-6}{(x-1)(x-2)(x-3)}$$
$$= \frac{3(x-2)}{(x-1)(x-2)(x-3)} = \frac{3}{(x-1)(x-3)}$$